PDE modeling in Life Sciences
Le Havre Normandie University
Master Degre

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Contents

Chapter 1

Some PDEs in biological context

1.1 Introduction

This course is devoted to the analysis of some partial differential equations arising in life sciences.

1.2 The Fischer-KPP Equation

The Fischer-KPP Equation writes:

$$u_t = u_{xx} + u(1-u), t \in \mathbb{R}^+, x \in \mathbb{R} \tag{1.1}$$

It is a typical reaction-diffusion equation. In (1.1), u_{xx} is the diffusion, while $u(1-u)$ is the reaction term. R.A. Fisher[1] proposed this equation to describe the spatial spread of an advantageous allele, and explored its traveling wave solutions-KPP equation. Kolmogorov, Pikovsky and Piskunov [46] proposed a mathematical analysis for the traveling waves solutions of the equation. For an accessible mathematical analysis, we refer to [63]. A detailed analysis of this equation will be provided in chapter 4.

Exercise 1.

1. Find constant in time solutions of (1.1).

[1]Ronald Fisher (or Sir Ronald Aylmer Fisher) born Feb 17 1890 in London- died, Jul 29 1962 in Adelaide, Australia, is highly recognized for his contributions in statistics such as the analysis of variance (ANOVA model) which is for example widely used in bio-statistics and biomedical applications. He was indeed the first to introduce the term of variance. He is mentioned in this course as the first to propose Equation (1.1) in the article The wave of advance of advantageous genes [30].

2. We look for solutions of (1.1) which write:

$$u(x,t) = v(x-ct)$$

with $c \in \mathbb{R}$ and v a function from $\mathbb{R}$ to $\mathbb{R}$. Prove that v must satisfy a second order differential equation (ODE). Write it as a two dimensional ODE of order 1.

1.3 Hodgkin-Huxley

Action potential propagation is a crucial phenomenon for information process in the nervous system and particularly in the brain. One of the paradigmatic models, which has been proposed to describe action potential propagation, is the Hodgkin-Huxley model. Written initially in 1952, see [40], for the description of the electrical activity of the squid giant axon, its formalism has served as basis of number of models widely used in Mathematical Neuroscience, see for example [3, 4, 5, 19, 31, 38, 49, 54] and references therein cited. At that time, Hodgkin and Huxley used the new voltage clamp technique to maintain constant the membrane potential. This technique allowed them to elaborate and fit the functional parameters of their model of four ODEs with their experiments. Basically, the model is obtained by considering the cell as an electrical circuit. The membrane acts as a capacitor whereas ionic currents result from ionic channels acting as variable voltage dependent resistances. The model takes into account three ionic currents: potassium (K^+), sodium (Na^+) and leakage (mainly chlorure, Cl^-). The Hodgkin-Huxley (HH) system reads as:

$$\left\{\begin{array}{rcl} CV_t & = & I + \overline{g}_{Na} m^3 h (E_{Na} - V) + \overline{g}_K n^4 (E_K - V) + \overline{g}_L (E_L - V) \\ n_t & = & \alpha_n(V)(1-n) - \beta_n(V) n \\ m_t & = & \alpha_m(V)(1-m) - \beta_m(V) m \\ h_t & = & \alpha_h(V)(1-h) - \beta_h(V) h, \end{array}\right. \tag{1.2}$$

where subscript t stands for derivative $\frac{d}{dt}$ and where I is the external membrane current, C is the membrane capacitance, $\overline{g}_i,\ E_i, i \in \{K, Na, L\}$ are respectively the maximal conductances and the (shifted) Nernst equilibrium potentials. The functions $\alpha(V)$ and $\beta(V)$ describe the transition rates between open and closed states of channels. They read as:

$$\begin{array}{rr} \alpha_n(V) = 0.01 \frac{-V+10}{\exp(1-0.1V)-1}, & \beta_n(V) = 0.125 \exp(-V/80), \\ \alpha_m(V) = 0.1 \frac{-V+25}{\exp(2.5-0.1V)-1}, & \beta_m(V) = 4 \exp(-V/18), \\ \alpha_h(V) = 0.07 \exp(-V/20), & \beta_h(V) = \frac{1}{1+\exp(-0.1V+3))}. \end{array} \tag{1.3}$$

The (shifted) Nernst equilibrium potentials are given by:

$$E_K = -12\ mV,\ \ E_{Na} = 120\ mV,\ \ E_L = 10.6\ mV$$

$$\overline{g}_K = 36\ mS/cm^2, \overline{g}_{Na} = 120\ mS/cm^2, \overline{g}_L = 0.3\ mS/cm^2.$$

These values are taken from [41], p 37-38, and correspond to those of the Hodgkin-Huxley original paper [40], after a change of variables $V = -V$. Recall that the Nernst equilibrium potentials are obtained by solving, for each ion $i \in \{K, Na, L\}$ the equation:

$$E_i = \frac{RT}{zF} \ln \frac{[i]_{out}}{[i]_{in}},$$

where $[i]_{in}$ and $[i]_{out}$ are concentrations of the ions inside and outside the cell. $R = 8.315$ is the universal gas constant, T is temperature in Kelvin, $F = 96, 48$ is the Faraday's constant, z is the valence of the ion. For example, this computation gives for sodium, with $T = 293$, $[i]_{out} = 440$, $[i]_{in} = 40$ (see [41] p 50):

$$E_{Na} \simeq 55,$$

which with a shift of $+65$ gives the value of 120 used here. Originally, Hodgkin and Huxley used the shift to obtain a potential at rest of approximately 0. Before going into more theoretical aspects, we give some interpretation about the form of conductances. The proportion of open potassium channels is n^4. This comes from the fact that 4 opening gates of potassium are required to open the potassium channel. Hence, the n gives the probability of the gate to be in active state and results in the n^4 term for potassium. For the sodium, it is supposed that there are there are three gates which open the channels and one which close them. Hence, the proportion of sodium opened channels is given by m^3h, where it is supposed that m stands for the probability of sodium opening gates to be active while h stands for probability of sodium closing gates to be active. For more details on various aspects of the HH model, we refer to [22, 25, 41] and the original paper [40].

The diffusion term-Equation of the cable

Up to now, we have only considered transversal current trough the membrane cell. We want now to include longitudinal current flowing through the axon. This paragraph is inspired from [1], see also [25]. First according to Ohm's law, the following equation holds:

$$V(x + dx, t) - V(x, t) = i(x, t)rdx$$

Dividing by dx and letting $dx \to 0$ leads to

$$V_x(x, t) = i(x, t)r$$

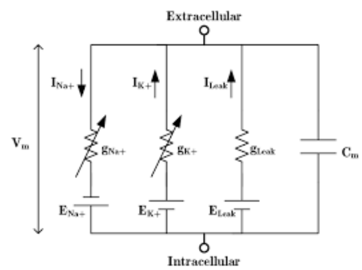

Figure 1.1: Electrical circuit used by Hodgkin and Huxley for modeling ionic fluxes trough channels

Then, according to Kirchoff law, between position x and $x + dx$, we have:

$$i(x + dx, t) = i(x, t) + (i_c + \sum_{ion} i_{ion}(x, t))dx.$$

Again, dividing by dx and letting $dx \to 0$ leads to

$$\frac{1}{r} V_{xx} = cV_t + \sum_{ion} i_{ion}.$$

Combining with the HH ODE model, we obtain the HH diffusive model:

$$\left\{ \begin{array}{rcl} cV_t &=& V_{xx} + I + \overline{g}_{Na} m^3 h(E_{Na} - V) + \overline{g}_K n^4 (E_K - V) + \overline{g}_L (E_L - V) \\ n_t &=& \alpha_n(V)(1 - n) - \beta_n(V)n \\ m_t &=& \alpha_m(V)(1 - m) - \beta_m(V)m \\ h_t &=& \alpha_h(V)(1 - h) - \beta_h(V)h, \end{array} \right. \tag{1.4}$$

Exercise 2.

1. We consider the diffusion less system. Show that, there exists V_m and V_M such that if n, m, h starts in $(0, 1)$ and V starts in (V_m, V_M), then they remain in this set for all positive time.

2. Show the same result for the diffusive system.

You can prove these results by a direct method or using the method of upper and lower solutions. Consider an ODE:

$$x' = f(x, t) \tag{1.5}$$

Definition 1. *A C^1 function x^+ is called an upper solution of* (1.5) *on* $[0, +\infty)$ *if it satisfies:*

$$(^+)' \geq f(x, t) \forall t \geq 0 \tag{1.6}$$

Proposition 1. *Assume that x^+ is an upper solution of* (1.5) *and x is a solution of* (1.5)*. Then, if $x^+(0) \geq x(0)$*

$$x^+(t) \geq x(t)\ \forall t \geq 0.$$

Furthermore, if $x^+(0) > x(0)$ then

$$x^+(t) > x(t)\ \forall t \geq 0.$$

Analog definitions and results hold for lower solutions. See [61]. A similar approach is useful for reaction-diffusion equations. Consider a reaction diffusion equation:

$$u' = f(u,t) + \Delta u \tag{1.7}$$

with Neumann Boundary Conditions.

Definition 2. *A regular function $u^+(x,t)$ is called an upper solution of* (1.7) *on $[0, +\infty$ if it satisfyes:*

$$u^{+'} - \Delta u \geq f(u,t) \forall t \geq 0 \tag{1.8}$$

Proposition 2. *Assume that u^+ is an upper solution of* (1.7) *and u is a solution of* (1.7)*, with NBC. Then, if $u^+(0) \geq u(0)$*

$$u^+(t) \geq u(t) \forall t \geq 0$$

For extended results around comparison of solutions in parabolic and elliptic problems, we refer for exampls to [55, 59].

1.4 FitzHugh-Nagumo

In 1961, R. FitzHugh proposed a $2D$ model that reproduces excitability and ocillatory features found in Hodgkin-Huxley model, see [31]. It is a modification of the well-known Van der Pol model, and has been initially called, the Bonhoeffer-van der Pol (BVP) model,

$$\begin{cases} x_t = c(F(x) + y + z) \\ y_t = \dfrac{1}{c}(x - a + by) \end{cases}$$

with,

$$w_t = \frac{dw}{dt},$$

and where F is a cubic function, $a, b > 0$, z corresponds to a stimulus intensity.

In the same paper [31], FitzHugh showed that the quantities $u = V - 36m$, $w = 0.5(n - h)$ obtained from the Hodkin-Huxley model evolve like the variables x and y of the BVP model. In 1962, Nagumo *et al.* proposed an electronic circuit whose behaviour is modeled by the BVP model, see [54]. The BVP model is now called the FitzHugh-Nagumo model. Another way to reduce the Hodgkin-Huxley to the FitzHugh-Nagumo model is to use properties of the Hodgkin-Huxley model and set, $h = 0.85 - n$ and $m(V) = \frac{\alpha_m(V)}{\alpha_m(V)+\beta_m(V)}$, then approximate the nullclines by a cubic and a straight line, see for example [41]. In this course, we will consider the following model of FitzHugh-Nagumo type,

$$\left\{ \begin{array}{rcl} \epsilon u_t &=& f(u) - v \\ v_t &=& u - \delta v - c \end{array} \right. \tag{1.9}$$

where

$$f(u) = -u^3 + 3u \text{ and } \epsilon > 0, \delta > 0 \text{ are small parameters.}$$

and its reaction-diffusion version:

$$\left\{ \begin{array}{rcl} \epsilon u_t &=& f(u) - v + d_u \Delta u \\ v_t &=& u - \delta v - c(x) + d_v \Delta v \end{array} \right. \tag{1.10}$$

with $u = u(x,t)$, $v = v(x,t)$, on a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with $d_u, d_v > 0$ and with Neumann zero flux conditions on the boundary Γ of Ω,

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0.$$

1.5 The Keller-Segel Equation

1.5.1 The original papers

Relying on observations of Escherichia coli placed in an environment with oxygen and an energy source, Keller and Segel [**?**] proposed a partial differential equation to model the apparition of bands in bacterial concentration traveling at constant speed in plates. The fundamental idea of the paper was to describe this phenomenon as the result of chemotaxis: the bacteria would avoid low concentrations and move preferentially toward higher concentrations of the substrate. Other effects taken into consideration were random motion (diffusion) and consumption of the substrate. No growth factor was considered. The orginal Keller-Segel [45] equation writes

$$\begin{cases} \frac{\partial b}{\partial t} & = \frac{\partial}{\partial x}(\mu(s)\frac{\partial b}{\partial x}) - \frac{\partial}{\partial x}(b\chi(s)\frac{\partial s}{\partial x}) \\ \frac{\partial s}{\partial t} & = -k(s)b + D\frac{\partial s}{\partial x^2} \end{cases} \tag{1.11}$$

These notations are those of the original article. Here b stands for the bacterial concentration and s for the substrate's concentration. The first term on the right of the first equation represents the motion of the bacteria in the absence of chemotaxis. The second term on the right side of the first equation describes the chemotactic response of the bacteria. It is assumed that the part of the bacterial flux which is the result of chemotaxis is proportional to the chemical gradient. This assumption is analog with assumptions used to derive the heat equation; (1.11) and related equations have been widely investigated theoretically and numerically. We refer for example to [**?**, 23, 52] for such developments.

Exercise 3. We consider (1.11) with $\xi(s) = \frac{1}{s}$, $k(s) = 1, \mu(s) = 1$, and $D = 0$, on the domain $\Omega = \mathbb{R}$ with the following assumptions

$$\lim_{x\to+\infty} b(x) = 0, \lim_{x\to+\infty} b'(x) = 0, \lim_{x\to+\infty} s(x) = s_\infty > 0, \lim_{x\to+\infty} s'(x) = 0.$$

Look for solutions of the form

$$b(x,t) = u(x - ct),\ s(x,t) = v(x - ct).$$

1.5.2 Neurodegerative diseases and angiogenesis

Neurodegenerative diseases(ND) are a major wordly burden. The most studied is Alzheimer's disease (AD). The first report on AD goes back to 1906, when Alois Alzheimer, a physician gave a lecture about the condition of a woman named August Deter at a Psychiatrists' research meeting in Tubingen, Germany. He related her decline after fifty years starting with including memory loss, cognitive impairment and hallucinations, followed by anxiety, desorientation, and sometimes delirium. It also included memory loss, cognitive impairment and hallucinations. She died within 5 years. After her death, Alzheimer examined her brain and identified a number of abnormalities, including thinning of the cerebral cortex, deposits of a amyloid plaques and neurofibrillary tangles at neuron's place. These last two observations would become be considered hallmarks of Alzheimer's disease. AD starts to attract the attention of mathematicians. Models derived from KS and FKPP can be used to investigate phenomena arising in ND.

1.6 The Fokker-Planck Equation

Consider a stochastic differential equation of the form

$$dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dW_t \tag{1.12}$$

where W_t stands for the Brownian motion. Then, the probability $p(x,t)$ to be at the position x at time t is given by the following parabolic equation, known as the Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(\mu(x,t)p(x,t)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x,t)p(x,t)) \tag{1.13}$$

Classical references are [58, 64]
Exercise 4.

1. Write the Fokker-Planck Equation associated to the SDE

$$dx_t = -adt + \sigma dW_t \tag{1.14}$$

2. Solve this equation.

3. Simulate the Fokker-Planck along with the statistical distribution from a large number of solutions of eq. (1.14).

Derivation of the Fokker-Planck Equation
See [53].

1.7 Fundamental solutions of Laplace and Poisson Equations

The Laplace equation

$$\Delta u = 0$$

and the Poisson equation

$$\Delta u = f$$

play important role in physics. For example they provide stationary solutions of the heat equation. In electrostatics, solving the Poisson equation amounts to finding the electric potential for a given charge distribution. In the context of this course, one can mention that recent developments of software tools for non-invasive brain stimulation rely on electrical potential provided by Poisson equations. From a mathematical point of view, the fundamental solutions are also crucial to develop the theory of PDEs in the L^p setting. The L^p setting is more technical than the L^2 setting. In this section we shall provide some results on solutions of Laplace and Poisson equations. The solutions of Laplace equations are called harmonic functions. We first recall important formulas. We start with the divergence formula

$$\int_{\Omega} \nabla \cdot f dx = \int_{\partial\Omega} f \cdot \nu ds$$

where ∇ stands for the divergence operator, $i.e.$ for a function $f : \mathbb{R}^n \to \mathbb{R}^n$

$$\nabla \cdot f = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i \cdot \nu ds,$$

and ν denotes the outward normal unit vector. From the divergence theorem applied to the gradient of a function $u : \mathbb{R}^n \to \mathbb{R}$ one deduce

$$\int_{\Omega} \Delta u dx = \int_{\Omega} \nabla \cdot \nabla u dx = \int_{\partial\Omega} \nabla u \cdot \nu.$$

We follow here [Eva10] and [Jos13].

Exercise

Find radial solutions of the Laplace equation for $n \geq 2$.

solution

We start with the case $n = 2$. We set

$$u(x) = \varphi(||x||),$$

with $x = (x_1, x_2)$.

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \varphi'(||x||) \times \frac{1}{2}(x_1^2 + x_2^2)^{-\frac{1}{2}} \times 2x_1 \\ &= \varphi'(||x||)(x_1^2 + x_2^2)^{-\frac{1}{2}} x_1.\end{aligned}$$

And,

$$\frac{\partial^2 u}{\partial x_1^2} = \varphi'(||x||)(x_1^2+x_2^2)^{-1}x_1^2 - \varphi'(||x||)(x_1^2+x_2^2)^{-\frac{3}{2}}x_1^2 + \varphi'(||x||)(x_1^2+x_2^2)^{-\frac{1}{2}}.$$

It follows that

$$\Delta u = \varphi'(||x||) + \frac{1}{||x||}\varphi'(||x||).$$

Denoting $||x|| = r$, we look, for a function φ such that

$$\varphi''(r) + \frac{1}{r}\varphi'(r) = 0.$$

This equivalent to

$$\frac{\varphi''(r)}{\varphi'(r)} = -\frac{1}{r}.$$

This gives

$$\ln(|\varphi'|) = -\ln(r) + c,$$

$$\varphi' = \frac{c}{r}, c \in \mathbb{R},$$

and therefore

$$\varphi(r) = c\ln(r) + c_2.$$

For $n \geq 3$ analog computations provide

$$\varphi(r) = cr^{2-n} + c_2.$$

In particular, we found that

$$\phi(x) = \begin{cases} \ln(||x||) & \text{if } n = 2 \\ ||x||^{2-n} & \text{if } n > 2 \end{cases}$$

is harmonic if $x \neq 0$.

Exercise 5. Prove that for $n \geq 2$, and for K compact

$$\int_K |\phi(x)|dx < +\infty$$

solution

It is sufficient to prove the result for $K = B(0,1)$. For $n = 2$ use polar coordinates, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. We find that

$$\int_{B(0,1)} |\ln(||x||)|dx_1dx_2$$

$$= -2\pi\int_0^1 r\ln r dr$$

$$= \frac{\pi}{2}.$$

For $n \geq 3$, we write

$$\int_{B(0,R)} \varphi(||x||)dx = \int_0^R \int_{\partial B(0,r)} \varphi(r)drd\sigma$$

$$= \int_0^R \int_{\partial B(0,1)} r^{n-1}\varphi(r)drd\sigma$$

$$= n\alpha(n)\int_0^R r^{n-1}\varphi(r)dr$$

$$= n\alpha(n)\int_0^R r^{n-1}r^{2-n}dr$$

$$= n\alpha(n)\frac{R^2}{2}$$

where $\alpha(n)$ denotes the volume of the unit sphere in $\mathbb{R}^n$, $\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$

Exercise

Prove that if $u, v \in C^2(\bar{\Omega})$,

$$\int_\Omega \Delta uv dx - \int_\Omega u\Delta v dx = \int_{\partial\Omega} v\nabla u.\vec{n}d\sigma(x) - \int_{\partial\Omega} u\nabla v.\vec{n}d\sigma \tag{1.15}$$

solution

Hint: Use Green formula

We set

$$\phi(x) = \begin{cases} \frac{1}{2\pi}\ln(||x||) & \text{if } n = 2 \\ \frac{1}{n(2-n)\alpha(n)}||x||^{2-n} & \text{if } n > 2 \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. If $u \in C^2(\bar{\Omega})$, then under the notations above, for $x \in \Omega$

$$u(x) = \int_{\partial\Omega}\Big(u(y)\frac{\partial\phi}{\partial\nu}(x-y) - \phi(x-y)\frac{\partial u}{\partial\nu}(y)\Big)dy + \int_\Omega \phi(x-y)\Delta u(y)dy \tag{1.16}$$

Exercise

Prove Section 1.7.

solution

Let $\epsilon > 0$ such that $B(x,\epsilon) \subset \Omega$. The idea is to apply Equation (1.15) with $u(y) = \phi(x-y)$ and $v(y) = u(y)$ on $\Omega \setminus B(x,\epsilon)$, then take the limit at $\epsilon = 0$.

$$\int_{\Omega\setminus B(x,\epsilon)}\Phi(x-y)\Delta u(y)dy - \int_{\Omega\setminus B(x,\epsilon)}\Delta\Phi(x-y)udy = \int_{\partial\Omega}\Phi\nabla u.\vec{n}d\sigma - \int_{\partial\Omega}u\nabla\Phi.\vec{n}d\sigma$$

$$-\int_{\partial B(x,\epsilon)}\Phi\nabla u.\vec{n}d\sigma + \int_{\partial B(x,\epsilon)}u\nabla\Phi.\vec{n}d\sigma$$

Then we remark that,

$$|\int_{\partial B(x,\epsilon)}\Phi\nabla u.\vec{n}d\sigma| \leq \int_{\partial B(x,\epsilon)}|\Phi|\nabla u|_\infty d\sigma$$

$$\le \int_{\partial B(x,\epsilon)} |\Phi| \nabla u|_\infty d\sigma$$
$$\le \varphi(\epsilon)|\nabla u|_\infty \epsilon^{n-1} \int_{\partial B(x,1)} d\sigma$$
$$\le \varphi(\epsilon)|\nabla u|_\infty \epsilon^{n-1} \int_{\partial B(x,1)} d\sigma$$
$$\to 0 \text{ as } \epsilon \to 0.$$

Also,

$$\int_{\partial B(x,\epsilon)} u\nabla\Phi(x-y).\vec{n} d\sigma = \int_{\partial B(x,\epsilon)} \varphi'(\epsilon)||\vec{n}||^2 u(y) d\sigma$$
$$= \varphi'(\epsilon)|B(x,\epsilon)| \frac{1}{|B(x,\epsilon)|} \int_{\partial B(x,\epsilon)} u(y) d\sigma$$
$$\to u(y) \text{ as } \epsilon \to 0.$$

This proves the result.

In the following results about harmonic functions, we closely follow [33] paragraphs 2.1 to 2.3. Two fundamental papers are [ADN59, ADN64]

1.7.1 Mean Value

Theorem 1. *Let $u \in C^2(\Omega)$ such that $\Delta u = 0$, then for any ball $B(y,R)$ strictly included in Ω*

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u(s) ds$$
$$u(y) = \frac{1}{\omega_n R^n} \int_B u(s) ds$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. If $\Delta u \ge 0$ $(\le)$ then the equalities are replaced by $\le$ $(\ge)$.

Theorem 2 (Strong Maximum Principle). *Let $u \in C^2(\Omega)$ such that $\Delta u \ge 0$, and suppose there exists a point $y \in \Omega$ such that*

$$u(y) = \sup_\Omega u.$$

Then u is constant. Analog result holds with $\Delta u \le 0$ and inf.

Theorem 3 (Weak Maximum Principle). *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that $\Delta u \ge 0$ and assume that Ω is bounded. Then*

$$\sup_{x\in\Omega} u = \sup_{x\in\partial\Omega} u.$$

Corollary 1. *Let $u \in C^2(\Omega) \cap \mathbb{C}^0(\bar{\Omega})$ such that $\Delta u = 0$ and assume that Ω is bounded. Then*

$$\inf_{x \in \partial\Omega} u \leq u \leq \sup_{x \in \partial\Omega} u.$$

Theorem 4 (Harnack Inequality)**.** *Let Ω' be a bounded domain strictly included in Ω. Let u be a non-negative function such that $\Delta u = 0$. Then*

$$\sup_{x \in \Omega'} u \leq C \inf_{x \in \Omega'} u.$$

Chapter 2

Bifurcations in Reaction-Diffusion systems

2.1 Turing

The idea behind the Turing mechanism can be formulated as follows: consider an ODE system with a stable stationary point:

$$\begin{cases} u_t &= f(u,v) \\ v_t &= g(u,v) \end{cases} \tag{2.1}$$

Can this stationary point become unstable if we add diffusion, in a system like:

$$\begin{cases} u_t &= f(u,v) + d_u \Delta u \\ v_t &= g(u,v) + d_v \Delta v \end{cases} \tag{2.2}$$

In this paragraph, we shall look at this in some detail. We consider the following system:

$$\begin{cases} u_t &= u - v \\ v_t &= 3u - 2v \end{cases} \tag{2.3}$$

Exercise 6. Determine the nature of the stationary point $(0,0)$.

We consider the following system:

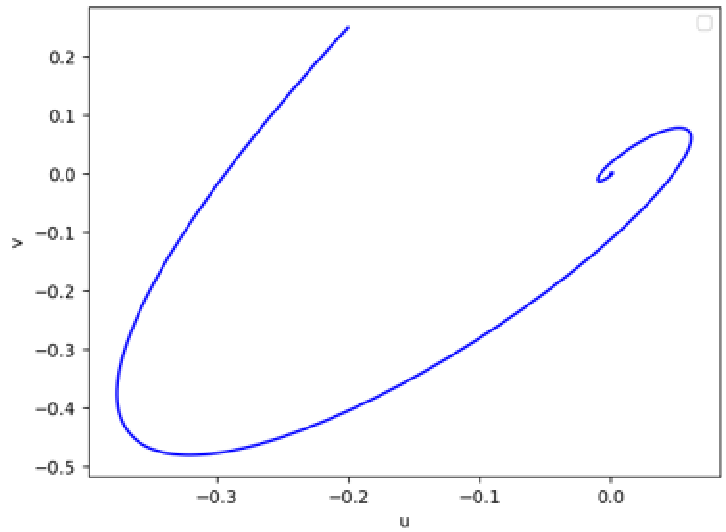

Figure 2.1: Trajectory associated to system (2.3). $(0, 0)$ is a stable focus.

$$\begin{cases} u_t = u - v + \sigma u_{xx} \\ v_t = 3u - 2v + v_{xx} \end{cases} \tag{2.4}$$

on the domain $(0, 1)$ with zero flux Neuman boundary conditions $u_x(0) = u_x(1) = 0$, and $\sigma > 0$.

Exercise 7.

1. Show that, in $L^2(0, 1)$ endowed with the scalar product $(f, g) = \int_0^1 f(x)g(x)dx$, the family $1, (\sqrt{2}(\cos(k\pi x))_{k\in\mathbb{N}^*}$ is orthonormal.

2. We set $\varphi_0(x) = 1$, and $\forall k \in \mathbb{N}^*\ \varphi_k(x) = \sqrt{2}\cos(k\pi x)$. Show that the functions φ_k satisfy:

$$-(\varphi_k)_{xx} = \lambda_k \varphi_k$$

and

$$(\varphi_k)_x(0) = (\varphi_k)_x(1) = 0,$$

where the values λ_k are to be specified.

3. We set,
$$u(t) = \sum_{k=0}^{\infty} u_k(t)\varphi_k,\ v(t) = \sum_{k=0}^{\infty} v_k(t)\varphi_k. \tag{2.5}$$
Show that for each $k \in \mathbb{N}^*$, u_k and v_k are solutions of a two dimensional linear system (write it). We denote this ODE system by (E_k).

4. Show that for fixed σ, $(0,0)$ is a sink for (E_k) if k is large enough.

5. Show that for each λ_k, there exists a value σ_k such that as σ crosses σ_k from right to left, the 2d ODE associated system E_k features a bifurcation from a sink to a saddle node.

6. Compute σ_1.

Solution
Computations lead to

$$\left\{ \begin{array}{lcl} u_{kt} & = & u_k - v_k - \sigma\lambda_k u_k \\ v_{kt} & = & 3u_k - 2v_k - \lambda_k v_k \end{array} \right. \tag{2.6}$$

and,

$$Tra(A_k) = -1 - (1+\sigma)\lambda_k < 0,\ Det(A_k) = \sigma\lambda_k^2 + 2\sigma\lambda_k + 1 - \lambda_k$$

Define σ_k as:
$$\sigma_k = \frac{\lambda_k - 1}{\lambda_k^2 + 2\lambda_k} > 0.$$

When σ crosses from right to left $Det(A_k)$ changes its sign from positive to negative. The following proposition follows. Define σ_k as:
$$\sigma_k = \frac{\lambda_k - 1}{\lambda_k^2 + 2\lambda_k} > 0.$$

The following proposition holds.

Proposition 3. *For each $k \in \mathbb{N}^*$, as σ crosses σ_k from right to left, the 2d ODE associated system E_k features a bifurcation from a sink to a saddle node. There is only a finite number of systems E_k for which $(0,0)$ is not a sink.*

Exercise 8. Prove rigorously that if (u_k, v_k) are solution of system (2.6) then (2.5) define solutions of (2.4).

The following lemma holds.

Lemma 1. *The sequence $(\sigma_k)_{k\in\mathbb{N}^*}$ is decreasing*

Therefore, since $||u||_{L_2} = \sum_{k=0} u_k^2$, the following proposition holds:

Proposition 4. *Suppose that $\sigma < \sigma_1$. Suppose that the initial condition is not the origin and that there exists one $k \in \mathbb{N}^*$ for which $(0,0)$ is a saddle for E_k and $(u_k(0), v_k(0))$ does not lie in the stable manifold associated with E_k, then*

$$\lim_{t\to+\infty} ||(u,v)(t)||_{L_2} = +\infty.$$

We have also:

Theorem 5. *Assume $\sigma > \sigma_1$, then for all IC*

$$\lim_{t\to+\infty} ||(u,v)(t)||_{L_2} = 0$$

Proof. From the above computations, for each $k \in \mathbb{N}^*$, the two eigenvalues of A_k write:

$$\mu_{1k} = \frac{\lambda_k}{2}\Big(-\frac{1}{\lambda_k} - (\sigma+1) - (\sigma-1)\sqrt{1+\epsilon(\frac{1}{\lambda_k})}\Big)$$

and

$$\mu_{2k} = \frac{\lambda_k}{2}\Big(-\frac{1}{\lambda_k} - (\sigma+1) + (\sigma-1)\sqrt{1+\epsilon(\frac{1}{\lambda_k})}\Big)$$

where $\epsilon()$ denotes a generic continuous function such that $\epsilon(0) = 0$. It follows that

$$\mu_{1k} = C_1 + \lambda_k(-\sigma + \epsilon(\frac{1}{\lambda_k}))$$

and

$$\mu_{2k} = C_2 + \lambda_k(-1 + \epsilon(\frac{1}{\lambda_k}))$$

where C_1 and C_2 are two constants. Note that for k large enough, the two eigenvalues are real, and uniformly smaller than a negative constant.
Let P_k the matrix of eigenvectors associated to μ_{1k} and μ_{2k}. Then

$$\begin{pmatrix} u_k(t) \\ v_k(t) \end{pmatrix} = P_k \begin{pmatrix} e^{-\mu_{1k}t} & 0 \\ 0 & e^{-\mu_{2k}t} \end{pmatrix} P_k^{-1} \begin{pmatrix} u_k(0) \\ v_k(0) \end{pmatrix}$$

Finally, there exists two constants δ ans C, such that:

$$||(u,v)(t)||_{L^2}^2 \leq Ce^{-\delta t}||(u,v)(0)||_{L^2}^2$$

□

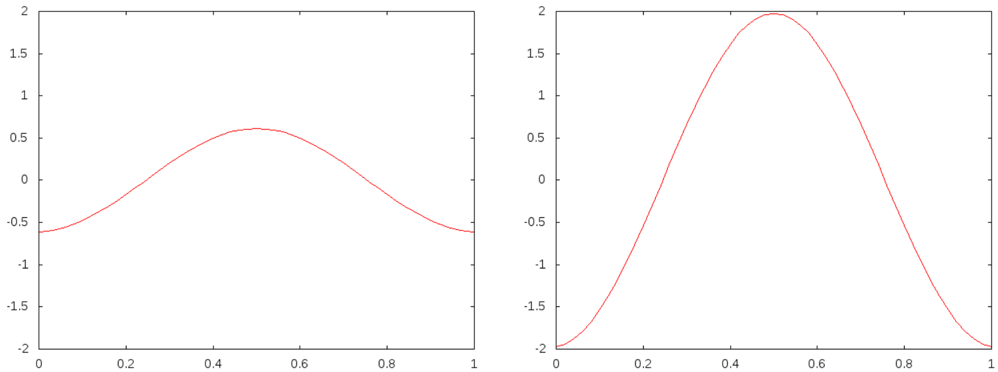

Figure 2.2: For $\sigma = 0.02$, the stationary solution $(0, 0)$ is unstable. Left: $t = 10$. Right $t = 17$.

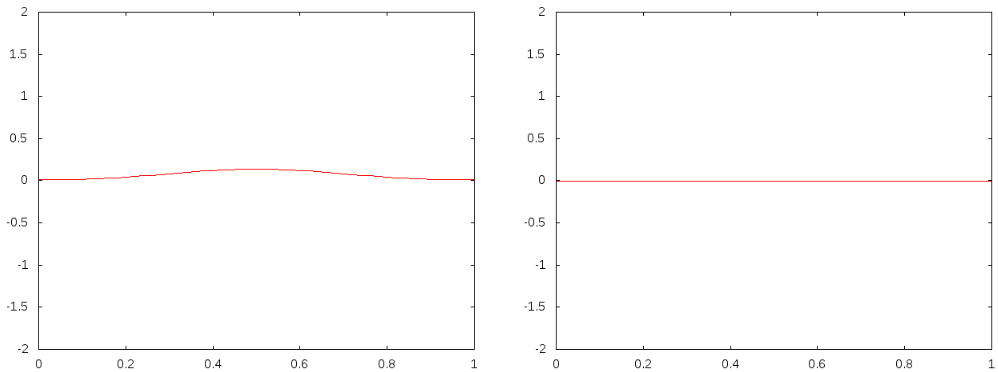

Figure 2.3: For $\sigma = 0.1$, the stationary solution $(0, 0)$ is stable. Left: $t = 0.1$. Right $t = 10$.

Exercise 9. Simulate equation (2.4). Exhibit numerically, two values of σ, for which $(0,0)$, appears to be respectively stable and unstable.

For the general system

$$\begin{cases} u_t &= au + bv + \sigma_u u_{xx} \\ v_t &= cu + dv + \sigma_v v_{xx} \end{cases} \tag{2.7}$$

on a regular bounded domain Ω and with Neumann zero flux boundary conditions, $\sigma_u > 0$, $\sigma_v > 0$, we have the following theorem. E_k is defined as in the exercise above.

Theorem 6. *We assume that $a > 0, d < 0$, $a + d < 0$ and $ad - bc > 0$. We fix $\sigma_v > 0$, then for each k large enough, there exists a value σ_{uk} such that as σ_u crosses σ_{uk} from right to left, the 2d ODE associated system E_k features a bifurcation from a sink to a saddle node. Furthermore for each value of σ, there is at most a finite number of systems E_k for which $(0,0)$ is not a sink.*

Exercise 10. Prove theorem 6 (generalize the proof sketched in the previous exercise).
Hint: It is well known that the eigenvalues of the operator $-\Delta u$ with Neuman Boundary conditions are an orthonormal basis of L^2 with associated discrete eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ which satisfy $\lim_{k\to+\infty}\lambda_k = +\infty$. You can directly use that.
Exercise 11.
We consider the system

$$\begin{cases} u_t &= -u^3 + u - v + \sigma u_{xx} \\ v_t &= 3u - 2v + v_{xx} \end{cases} \tag{2.8}$$

on a regular bounded domain Ω and with Neumann zero flux boundary conditions.

1. Compute the linearized system around $(0,0)$. What do you remark?

2. Simulate system (3.1).

2.2 Hopf

Usually, in 2d nonlinear systems, Hopf bifurcation relates to the case in which an eigenvalue of the Jacobian crosses the imaginary case, and this gives raise to the coexistence of a stationary state of spiral type and a limit cycle. Since our goal is to focus on PDEs, we start with linear systems which provide examples where explicit computations can be provided. However, in 2d linear systems, limit cycles correspond only to centers. We consider the following system:

$$\begin{cases} u_t &= \alpha u - v \\ v_t &= u \end{cases} \tag{2.9}$$

Exercise 12. Discuss the nature and stability of the stationary point of (2.9).

Following the scheme of the previous section, we move to a RD system. We consider:

$$\begin{cases} u_t &= \alpha u - v + u_{xx} \\ v_t &= u \end{cases} \tag{2.10}$$

on the domain $(0, 1)$ with Neumann Boundary conditions.
Exercise 13.
Analyze the system (2.10).
Solution
As previously, we write

$$u(t) = \sum_{k=0}^{\infty} u_k(t)\varphi_k,\ v(t) = \sum_{k=0}^{\infty} v_k(t)\varphi_k.$$

and plug these expressions into Equation (2.10). We obtain,

$$(E_k) \begin{cases} u_{kt} &= \alpha u_k - v_k - \lambda_k u_k \\ v_{kt} &= u_k \end{cases} \tag{2.11}$$

With the notations of the previous section, we have:

Proposition 5. *For* $\alpha < 0$*,*

$$\lim_{t \to +\infty} ||(u, v)(t)||_{L^2} = 0$$

Proposition 6. *Let* $k \in \mathbb{N}^*$.
For $\alpha = \lambda_k$, $(0,0)$ *is a center for system* (E_k), *a source for* (E_l) *if* $l < k$ *and a sink for* (E_l) *if* $l > k$. *Furthermore, if:* $u_l(0) = v_l(0) = 0$ *for* $l \in \{0, ..., k-1\}$ *then*

$$\lim_{t\to+\infty} ||(u,v)(t) - \varphi_k(u_k(t), v_k(t))|| = 0.$$

Otherwise,

$$\lim_{t\to+\infty} ||(u,v)(t)|| = +\infty.$$

For $\lambda_k < \alpha < \lambda_{k+1}$, $(0,0)$ *is a source for* (E_l) *si* $l \leq k$ *and a sink for* (E_l) *if* $l > k$. *Furthermore, if* $u_l(0) = v_l(0) = 0$ *for* $l \in \{1, ..., k\}$ *then*

$$\lim_{t\to+\infty} ||(u,v)(t)|| = 0.$$

Otherwise

$$\lim_{t\to+\infty} ||(u,v)(t)|| = +\infty.$$

This proposition is valid for any dimension replacing $k^2\pi^2$ and $\cos(k\pi x)$ by the eigenvalues and eigenfunctions of the laplacian operator with Neuman Boundary conditions.

Exercise 14.
Illustrate proposition 5 with numerical simulations. What do you remark?

Exercise 15.
Using the theoretical results of proposition 5, write a code allowing to illustrate properly special solutions of (2.10) We now consider the system

$$\left\{ \begin{array}{rcl} u_t &=& \alpha u - u^3 - v + u_{xx} \\ v_t &=& u \end{array} \right. \tag{2.12}$$

on the domain $(0,1)$ with Neumann Boundary conditions.

Exercise 16.
Analyze the system

$$\left\{ \begin{array}{rcl} u_t &=& \alpha u - v - u(u^2+v^2) \\ v_t &=& u + \alpha v - v(u^2+v^2) \end{array} \right. \tag{2.13}$$

and

$$
\begin{cases} u_t = \alpha u - v + u(u^2+v^2) \\ v_t = u + \alpha v + v(u^2+v^2) \end{cases} \tag{2.14}
$$

Exercise 17.
Here we consider the ODE version of (2.12). We assume $|\alpha| < 2$.

1. Compute an eigenvector p associated to an eigenvalue λ of the jacobian A at $(0,0)$.
2. Compute an eigenvector q of A^t associated to $\bar{\lambda}$ such that $< p, q >= 1$ where $<,>$ denotes the scalar product in $\mathbb{C}^2$.
3. Prove that we can write $(u,v) = zq + z\bar{q}$.
4. Show that $< p, \bar{q} >= 0$.
5. Show that $< p, (u,v) >= z$.
6. Provide the differential equation satisfied by z.
7. Show that after a change of variables this equation writes

$$w' = \lambda w - 3\bar{p}_1 w^2 \bar{w} + O(|w|^4).$$

Exercise 18.
Show that the ODE version of (2.12) admits a limit-cycle.
Exercise 19.
Simulate (2.12)
Exercise 20.
Apply the procedure of the previous exercise to the PDE. Prove the existence of the center manifold.

Chapter 3

Patterns

3.1 Numerical simulations

Exercise 21.
We consider the system

$$\left\{\begin{array}{rcl} u_t &=& -u^3+u-v+\sigma u_{xx} \\ v_t &=& 3u-2v+v_{xx} \end{array}\right. \tag{3.1}$$

on the domain $\Omega=(0,a)\times(0,a)$ and with Neumann zero flux boundary conditions. Simulate system (3.1) and try to exhibit patterns.

Exercise 22.
Same question with the system

$$\left\{\begin{array}{rcl} u_t &=& -u^3+3u-v+\sigma u_{xx} \\ v_t &=& u \end{array}\right. \tag{3.2}$$

3.2 Analysis

Since the spectrum of the laplacian plays an important rôle, we are going to spend some time to describe its properties in higher dimensions. This will be the occasion to review some topological results.

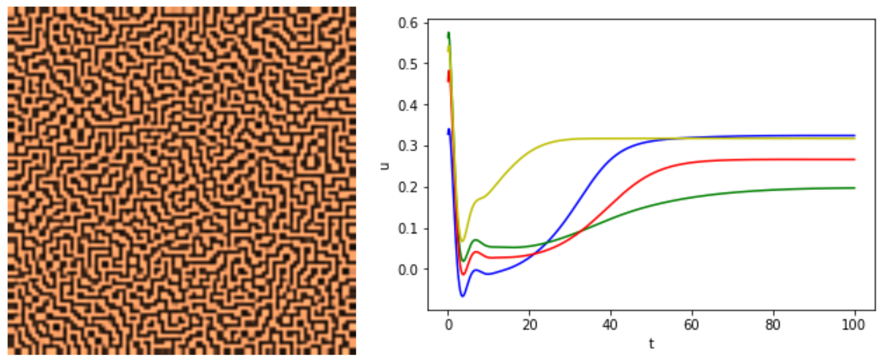

Figure 3.1: Pattern obtained from simulation of equation (3.1) with $\sigma = 0.1$. Left: isovalues of $u(x,t)$ at the final time $t = 100$. Right: time evolution of $u(x,t)$ for $t \in (0, 100)$ at four fixed space coordinates.

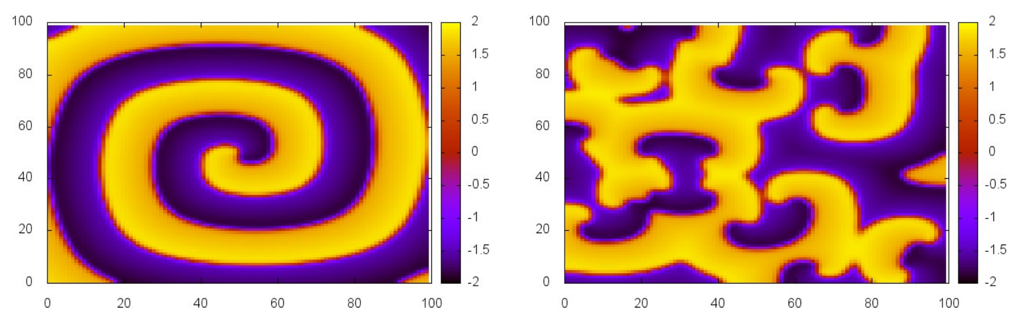

Figure 3.2: Patterns obtained from simulation of equation (3.2)

3.2.1 Eigenvalues in the square

Exercise 23. Let $\Omega = (0, a) \times (0, a)$. Exhibit an Hilbertian basis of eigenfunctions of the operator $-\Delta$ with Neumann boundary conditions of the space $L^2((0, a) \times (0, a))$.

3.2.2 Generalization: spectral properties of the Laplacian operator

We focus on the spectral properties of the Laplacian operator with NBC. We follow the exposition of [42], section 11.5. We assume that $\Omega \subset \mathbb{R}^n$ is an open bounded set with regular boundary and use the following notations

$$\mathcal{H} = L^2(\Omega),\ \mathcal{V} = H^1(\Omega).$$

We look for λ and u solutions of

$$-\Delta u = \lambda u \tag{3.3}$$

with $\nabla u.\nu = 0$ on the boundary of Ω, ν denoting the outward unitary vector at the boundary.

First of all, let us remark that $\lambda_0 = 0$ is an eigenvalue of the Laplacian operator associated with the constant eigenfunction

$$\varphi_0(x) = \frac{1}{\left(\int_\Omega dx\right)^{\frac{1}{2}}}.$$

Next, we look for another eigenvalue as follows. Let

$$\lambda_1 = \inf_{u \in \mathcal{V}, ||u||=1, \int u dx = 0} ||\nabla u||^2$$

where

$$||u||^2 = \int_\Omega u^2 dx \text{ and } ||\nabla u||^2 = \sum_{i=1}^{n} \int_\Omega u_{x_i}^2 dx.$$

Let (u_n) such that, $||u_n|| = 1$, $\int u_n dx = 0$ and,

$$\lim_{n \to +\infty} ||\nabla u_n||^2 = \lambda_1.$$

Since $\mathcal{V}$ is compactly embedded in $\mathcal{H}$, there exists a subsequence of (u_n) which we still denote by (u_n) which converges toward some φ_1 in $\mathcal{H}$, with $||\varphi_1|| = 1$. Next, we remark that

$$||\nabla u_n + \nabla u_m||^2 + ||\nabla u_n - \nabla u_m||^2 = 2||\nabla u_n||^2 + 2||\nabla u_m||^2$$

and therefore

$$||\nabla u_n - \nabla u_m||^2 \leq 2||\nabla u_n||^2 + 2||\nabla u_m||^2 - ||\nabla u_n + \nabla u_m||^2.$$

But by definition

$$\lambda_1 \leq \frac{||\nabla(u_n + u_m)||^2}{||u_n + u_m||^2},$$

which implies that

$$||\nabla u_n - \nabla u_m||^2 \leq 2||\nabla u_n||^2 + 2||\nabla u_m||^2 - \lambda_1 ||u_n + u_m||^2$$

$$\leq 2||\nabla u_n||^2 + 2||\nabla u_m||^2 - \lambda_1 \big(||u_n||^2 + ||u_m||^2 + 2(u_n, u_m)\big).$$

But since for n, m large enough (u_n, u_m) is arbitrary close to 1, the right-hand side is arbitrary close to zero for n, m large enough. It follows that (u_n) is a Cauchy sequence in $\mathcal{V}$ (endowed with its scalar product). Therefore (u_n) converges toward φ_1 in $\mathcal{V}$ and $\lambda_1 = |\nabla \varphi_1||^2$. Now, we want to prove that λ_1 and φ_1 satisfy Equation (3.3). Let $H_1 = \{f \in \mathcal{V}, \int f = 0\}$, and $\varphi \in H_1$, then for t restricted to a small neighborhood of 0

$$\lambda_1 = \inf_t \frac{||\nabla(\varphi_1 + t\varphi)||^2}{||\varphi_1 + t\varphi||^2}.$$

Therefore the $t-$derivative of

$$\frac{||\nabla(\varphi_1 + t\varphi)||^2}{||\varphi_1 + t\varphi||^2}$$

cancels at $t = 0$. Now, at $t = 0$

$$\frac{\partial}{\partial t}||\nabla(\varphi_1 + t\varphi)||^2|_{t=0} = 2\int \nabla\varphi_1.\nabla\varphi$$

and

$$\frac{\partial}{\partial t}||\varphi_1 + t\varphi||^2|_{t=0} = 2\int \varphi_1\varphi.$$

Therefore, we obtain

$$0 = 2||\varphi_1||^2 \int \nabla u.\nabla\varphi - 2||\nabla\varphi_1||^2 \int \varphi_1\varphi.$$

Since $||\varphi_1||^2 = 1$ and $||\nabla\varphi_1||^2 = \lambda_1$, this gives

$$\int \nabla\varphi_1 \nabla\varphi = \lambda_1 \int \varphi_1\varphi.$$

Note that this equality holds also for any $\varphi \in \mathcal{V}$ since it holds for any constant function φ. Regularity results for elliptic equations show that if $\partial\Omega$ is C^∞ then $\varphi_1 \in C^\infty(\bar{\Omega})$. By integration by parts, this gives

$$-\int \Delta\varphi_1\varphi + \int \nabla\varphi_1.\nu = \int \varphi_1\varphi$$

but since $-\Delta\varphi_1 = \lambda_1\varphi_1$, we obtain

$$\int \varphi\nabla\varphi_1.\nu = 0$$

which implies

$$\nabla\varphi_1.\nu = 0 \text{ on } \Omega.$$

By induction, one can construct a sequence of eigenfunctions, and eigenvalues. The following theorem states the result more precisely.

Theorem 7. *Let $\Omega \subset \mathbb{R}^n$ be an open,connected and bounded set of class C^∞. Then the equation*

$$-\Delta u = \lambda u,\ u \in \mathcal{V}$$

admits an infinite countable number of eigenvalues λ_k and associated eigenfunctions φ_k, as solutions. Furthermore

$$0 = \lambda_0 \leq \lambda_1 \leq ... \leq \lambda_k \leq ...$$

with

$$\lim_{k\to+\infty} \lambda_k = +\infty,$$

$$\int_\Omega \varphi_k\varphi_l = 0 \textit{ if } k \neq l,\ \int_\Omega \varphi_k^2 = 1,$$

$$\nabla\varphi_k.n = 0 \textit{ on } \partial\Omega.$$

Proof. For $k \in \{0,1\}$, λ_k and φ_k have been defined above. Assuming that they have been defined until an arbitrary order $k-1$, let

$$H_k = \{f \in \mathcal{V}, \int f = 0, \int f\varphi_1 = 0, ..., \int f\varphi_{k-1} = 0\}$$

and

$$\lambda_k = \inf_{u\in H_k, ||u||=1} ||\nabla u||^2,$$

then proceeding as above we can find $\varphi_k \in K_k$ with $||\varphi_k|| = 1$ and

$$-\Delta\varphi_k = \lambda_k\varphi_k$$

with

$$\nabla\varphi_k.\nu = 0.$$

We remark that since (H_k) is a decreasing sequence, the sequence (λ_k) is increasing. Next, we prove that the sequence (λ_n) as defined above satisfies

$$\lim_{n\to+\infty}\lambda_n = +\infty.$$

Assume that it is not the case. Then, there exists some constant C such that

$$||\nabla\varphi_k|| \leq C.$$

Again, by the compact injection from H^1 into L^2, one can extract a subsequence φ_k which converges in L^2. Therefore this subsequence is Cauchy in L_2. Note however, that for any φ_k, φ_l

$$||\varphi_k - \varphi_l||^2 = ||\varphi_k||^2 + ||\varphi_l||^2 - 2(\varphi_k, \varphi_l) = 2,$$

which contradicts the fact that (φ_k) is Cauchy.

Now, let $v \in \mathcal{V}$. Let $v_i = \varphi_i$. We shall prove that the series $\sum_{i=0}^{\infty} v_i\varphi_i$ converges to v in $\mathcal{H}$. Let $v^m = \sum_{i=0}^{m} v_i\varphi_i$ and let $w^m = v - v^m$. Note that for all $i \in \{1, ..., m\}$,

$$\int(v - v^m)\varphi_i = \int v\varphi_i - \int v\varphi_i = 0,$$

which means that w^m is orthogonal to $span\{\varphi_0, ..., \varphi_m\}$ and that v^m is the orthogonal projection of v on $span\{\varphi_0, ..., \varphi_m\}$. Therefore, by definition of λ_{m+1},

$$\lambda_{m+1} \leq \frac{||\nabla w^m||^2}{||w^m||^2},$$

which in turn implies

$$||w^m||^2 \leq \frac{1}{\lambda_{m+1}}||\nabla w^m||^2.$$

But

$$||\nabla w^m||^2 = ||\nabla v||^2 + ||\nabla v^m||^2 - 2\int \nabla v.\nabla v^m$$

$$= ||\nabla v||^2 - ||\nabla v^m||^2 - 2\int \nabla v.\nabla v^m + 2||\nabla v^m||^2$$

$$= ||\nabla v||^2 - ||\nabla v^m||^2 - 2\int \nabla(v - v^m).\nabla v^m$$

$$= ||\nabla v||^2 - ||\nabla v^m||^2.$$

It follows that

$$||w^m||^2 \leq \frac{1}{\lambda_{m+1}} ||\nabla v||^2 \xrightarrow[m \to +\infty]{} 0.$$

Finally, this proves the theorem thanks to classical resuts on Hilbertian sums (see for example [11] chapter 5) since $\mathcal{V}$ is dense in $\mathcal{H}$.

□

Chapter 4

Traveling Waves

4.1 Analysis of the FKPP equation

In this chapter, we will study in detail the Fisher-KPP equation. It is a good example to introduce the field of traveling waves analysis.

The Fischer-KPP Equation writes:

$$u_t = u_{xx} + u(1-u), t \in \mathbb{R}^+, x \in \mathbb{R} \tag{4.1}$$

We look for solutions of the form:

$$u(x,t) = w(x-ct).$$

Exercise 24. Show that w is a solution of a second order nonlinear ordinary differential equation (write it!)
Solution
We find:

$$w" = -cw' - w(1-w).$$

Exercise 25. Write the above second order ODE as a two dimensionnal first order ODE.
Solution
We find:

$$\left\{ \begin{array}{rcl} w' & = & p \\ p' & = & -cp - w(1-w) \end{array} \right. \tag{4.2}$$

By the Cauchy-Lipschitz theorem, the above equation admits a unique solution.
Exercise 26.

1. Find the stationary solutions of the above equation.
2. Proceed to a linear stability analysis of each of these solutions.

Solution

1. The two stationary solutions are $(0,0)$ and $(1,0)$.
2. The jacobian is given by:

$$J = \begin{pmatrix} 0 & 1 \\ 2w-1 & -c \end{pmatrix}$$

Therefore at $(0,0)$ the jacobian is

$$J_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$$

and its eigenvalues are

$$\frac{-c-\sqrt{c^2-4}}{2}, \quad \frac{-c+\sqrt{c^2-4}}{2}.$$

At $(1,0)$ the jacobian is

$$J_{(0,1)} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$$

and its eigenvalues are

$$\frac{-c-\sqrt{c^2+4}}{2}, \quad \frac{-c+\sqrt{c^2+4}}{2}.$$

The following theorem holds.

Theorem 8. *For $c \in (-\infty, 0)$, $(0,0)$ is a source. For $c \in (0,+\infty)$ it is a sink. For $c \in (-\infty,-2)$, it is an unstable node, while for $c \in (-2,0)$ it is an unstable focus. For $c \in (0,2)$ it is a stable focus. For $c \in (2,+\infty)$ it is an stable node.*
For all $c \in \mathbb{R}$, the stationary point $(1,0)$ is a saddle node.

The next step is to provide a relevant numerical analysis to gain insights on the global behavior. We want to emphasize here the importance to reconnect to the meaning of solutions of Equation (4.2) with respect to the original equation Equation (4.1) **Exercise 27.** Provide a numerical analysis.

Exercise 28. Draw the nullcines and, when relevant, the tangent attractive and repulsive manifolds of the fixed points in the cases $c = 1, c = 4$.

Solution

From the above computations, it follows that an eigenvector associated with an eigenvalue λ is given by

$$\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

Therefore, for $c = 1$, at $(1, 0)$, the eigenvalues are:

$$\frac{-1-\sqrt{5}}{2} \simeq -1.618, \text{ and } \frac{-1+\sqrt{5}}{2} \simeq 0.618.$$

It follows that at $(1, 0)$ the attractive manifold is tangential to the vector

$$\begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ -1.618 \end{pmatrix}$$

while the repulsive manifold is tangential to the vector

$$\begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ 0.618 \end{pmatrix}$$

$(0, 0)$ is a stable focus.

For $c = 4$, at $(0, 0)$, the eigenvalues are:

$$-2-\sqrt{3} \simeq -3.732, \text{ and } -2+\sqrt{3} \simeq -0.268.$$

It follows that at $(0, 0)$ the trajectories are tangential for large t to the vector

$$\begin{pmatrix} 1 \\ -2+\sqrt{3} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ -0.268 \end{pmatrix}$$

For $c = 4$, at $(1, 0)$, the eigenvalues are:

$$-2-\sqrt{5} \simeq -4.236, \text{ and } -2+\sqrt{5} \simeq 0.236.$$

It follows that at $(1, 0)$ the repulsive manifold is tangential to the vector

$$\begin{pmatrix} 1 \\ -2+\sqrt{5} \end{pmatrix} \simeq \begin{pmatrix} 1 \\ 0.236 \end{pmatrix}$$

Exercise 29. Determine whether the Hopf-Bifurcation is supercritical or undercritical

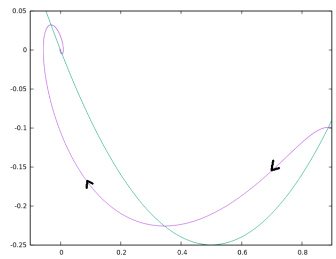

Figure 4.1: In purple, solution of system for $c = 1$ and IC $(0.9, -0.1)$. The stationary solution $(0, 0)$ is a stable focus. In green, the p-nullcline: $p = w(w - 1)$s.

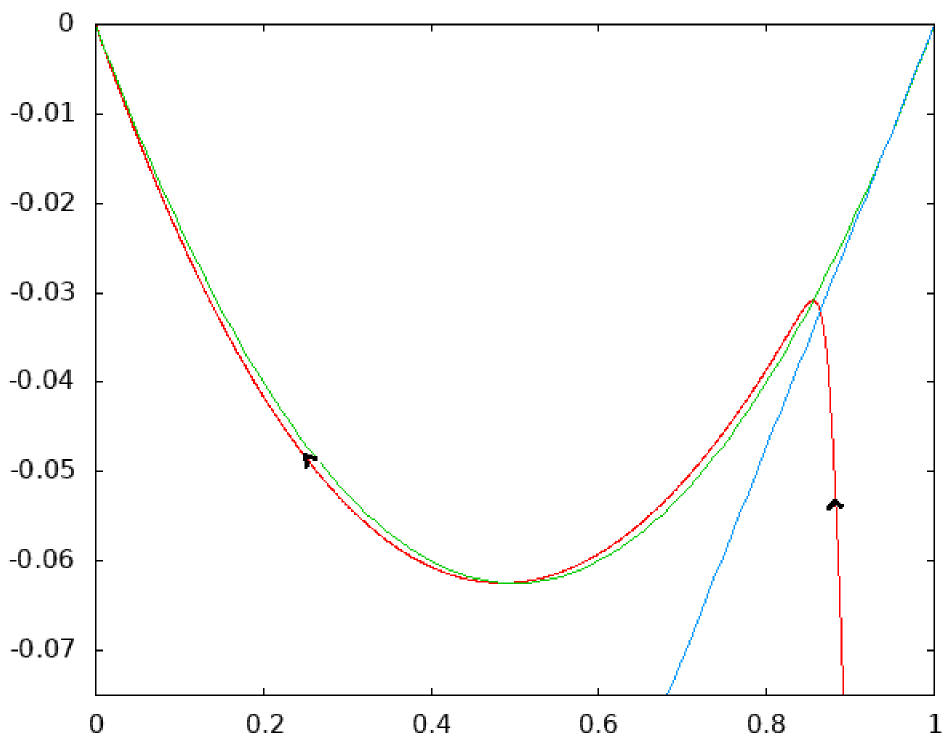

Figure 4.2: In red, solution of system for $c = 4$ and IC $(0.9, -0.1)$. The stationary solution $(0, 0)$ is a stable node. In green, the p-nullcline: $p = 0.25w(w - 1)$. In blue the repulsive manifold of the linearized system at $(1, 0)$

Exercise 30. Exhibit numerically an heteroclinic orbit for $c = 1$ and $c = 4$.
Exercise 31. Prove the above mentioned result for $c = 4$.

We state the result of this exercise as a theorem.

Theorem 9. *For $c > 2$, there exists an heteroclinic orbit from $(1, 0)$ to $(0, 0)$.*

Proof. Let $f(w) = w(1 - w)$. First, we remark that

$$\frac{\partial p}{\partial w} = -c - \frac{f(w)}{p}$$

Next, we claim that any solution of the equation

$$p'(w) = -c - \frac{1}{p(w)} f(w) \tag{4.3}$$

with $p(0) < 0$, $|p(0)|$ small, satisfies $p_(w) < -w$ on $[0, 1]$. Indeed, $p_2(w) = -w$ is an upper solution of (4.3). To see this, observe that

$$p_2(w) = -w$$

satisfies

$$p_2(0) = 0$$

and that

$$p_2'(w) = -1 > -1 - w = -2 + 1 - w = -2 + \frac{f(w)}{w} = -2 - \frac{f(w)}{-w}.$$

But since $c > 2$ and $p_2(w) = -w$, we obtain that

$$p_2'(w) > -c - \frac{f(w)}{p_2}.$$

It follows that p_2 is an upper solution. From this, reasoning with the nullclines, we deduce that the trajectory ensued from $(1, 0)$ at $t = -\infty$ and evolving downward will cross the nullcline $p = -\frac{1}{c} f(w)$ and then remain stuck between the line $w = 0$ and the nullcline. Together with the fact that $p' > 0$, $w' < 0$ in this region this implies that this trajectory converges toward $(0, 0)$ at $t = +\infty$. We give also a direct proof of the fact that a solution p with $p(0) < 0$ satisfies $p2(w) > p(w)$ on $[0, 1]$. Assume the existence of $w_1 \in (0, 1]$ such that

$$p(w) < p_2(w) \text{ on } [0, w_1)$$

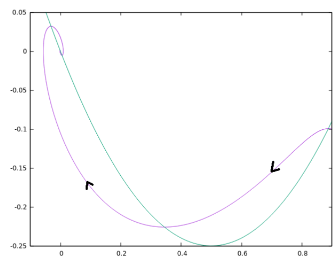

Figure 4.3: In purple, solution of system for $c = 1$ and IC $(0.9, -0.1)$. The stationary solution $(0, 0)$ is a stable focus. In green, the p-nullcline: $p = w(w - 1)$s.

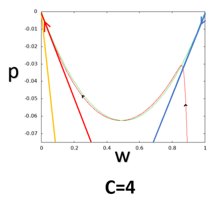

Figure 4.4: Heteroclin orbit

$$p(w_1) = p_2(w_1)$$

Let w_0 in $[0, w_1)$. We have

$$p(w_1) - p(w_0) = -c(w_1 - w_0) - \int_{w_0}^{w_1} (\frac{f(w)}{p_(w)}dw$$

$$p_2(w_1) - p_2(w_0) > -c(w_1 - w_0) - \int_{w_0}^{w_1} (\frac{f(w)}{p_2(w)}dw$$

Substracting the first equation to the inequation, we obtain:

$$-p_2(w_0) + p(w_0) > \int_{w_0}^{w_1} (-\frac{f(w)}{p_2(w)} + \frac{f(w)}{p(w)})dw > 0$$

which is a contradiction since $p_2(w) > p(w)$ on (w_0, w_1). $\square$

4.2 Traveling waves in the FHN equations

For the reader interested in the traveling waves in the Neuroscience context, we provide here some references for the study of this phenomenon in the FHN equation. Indeed, since the first studies on the Fisher-KPP equation [39, **?**, 51], the topic has aroused a huge interest, see for example [63]. The particular case of the wave propagation phenomenon in the diffusive FHN, with $\Omega = \mathbb{R}$ has also been intensively studied for a few decades. For example, in [56, 57], J. Rinzel and coauthors, following the ideas in McKean [50] studied a FHN RD system with piecewise linear nonlinearity (instead of the cubic nonlinearity). They provided explicit computations of periodic solutions, pulses, propagation speeds, and stability results. Around the same period, in a series of four articles, J.W. Evans provided a more theoretical analysis of a general model of nerve conduction, see [26, 27, 28, 29]. In [27], he defines the so-called Evans function, which would become an essential tool for traveling wave stability analysis see [9]. A few years later, see [43], C. Jones, relying on the papers of Evans, provided a detailed analysis focusing on the stability of the traveling waves of the FHN diffusive equation. Since then, many studies have been devoted to the characterization of different properties of traveling waves. This includes the proof of the existence of pulses of monotonic and periodic tales, the construction of solutions in a slow-fast context thanks to singular perturbation theory or via asymptotic expansions, etc..., see [14, 37, 13, 16, 15, 17, 44, 34, 35, 18, 47] and references therein cited.

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