HOPF BIFURCATION IN A OSCILLATORY-EXCITABLE REACTION-DIFFUSION MODEL WITH SPATIAL HETEROGENEITY

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Abstract

We focus on the qualitative analysis of a reaction-diffusion with spatial heterogeneity near a bifurcation. The system is a generalization of the well known FitzHugh-Nagumo system in which the excitability parameter is space dependent. This heterogeneity allows to exhibit concomitant stationary and oscillatory phenomena. We prove the existence of an Hopf bifurcation and determine an equation of the center-manifold in which the solution asymptotically evolves. Numerical simulations illustrate the phenomenon.

1 Introduction

The following reaction-diffusion system of FitzHugh-Nagumo (FHN) type:

$$\left\{\begin{array}{rcl} \epsilon u_t &=& f(u) - v + d_u \Delta u \\ v_t &=& u - c(x,t) - \delta v + d_v \Delta v \end{array}\right. \tag{1}$$

where $f(u) = -u^3 + 3u$, $\epsilon > 0$ small, $\delta \geq 0$, $c(x)$ regular function, $d_u \geq 0$, $d_v \geq 0$, $d_u d_v \neq 0$, and with Neumann Boundary (NBC) conditions on a regular bounded domain Ω, is relevant for obtaining different kind of patterns and interesting phenomena in physiological context. A property of system (1) is that, due to the dependence of c on space variable x, it can take advantage of both excitability and oscillatory regimes of the FHN system. Therefore, interesting phenomena can be obtained with this single Partial Differential equation such as spirals, mixed mode oscillations (MMO's), propagation of bursting oscillations, see [Ambrosio & Francoise(2009)]. Recall that the FitzHugh-Nagumo model, widely used in mathematical neuroscience, is obtained by a reduction of the Hodgkin-Huxley model (4 equations) awarded by the 1963 Nobel prize of Physiology and Medicine, see [FitzHugh(1961), Hodgkin & Huxley(1952), Nagumo & al.(1962)] for original papers or for example [Izhikevich(2005), Ermentrout & Termam(2010)] for good fundamental books. In this article, we focus on equation (1) in the case where c is only depending on x, $\delta = d_v = 0, d_u = d$, and the space dimension is 1, i.e.:

$$\left\{ \begin{array}{rcl} \epsilon u_t & = & f(u) - v + d u_{xx} \\ v_t & = & u - c(x) \end{array} \right. \tag{2}$$

on a real interval $\Omega =]-a, a[, a > 0$ and with NBC $u'(-a) = u'(a) = 0$. In order to understand the qualitative behavior of system (2), we must recall the behavior of the underlying ODE system:

$$\left\{ \begin{array}{rcl} \epsilon u_t & = & f(u) - v \\ v_t & = & u - c. \end{array} \right. \tag{3}$$

We have for appropriate values of parameters, the following theorem see [Ambrosio(2009)] and references therein, which is illustrated in figure 1

Theorem 1. *There exists a unique stationary point. If $|c| \geq 1$ the stationary point is globally asymptotically stable, whereas if $|c| > 1$, it is unstable and there exists a unique limit-cycle that attracts all the non constant trajectories. Furthermore, at $|c| = 1$, there is a supercritical Hopf bifurcation.*

Another important feature of system (3) is excitability: for $|c| > 1$ and c not so far from -1, if a solution is taken away of a certain region it undergoes a large oscillation before returning to its stable state. This can be well understood by slow-fast analysis. Typical behavior is represented in figure 1.

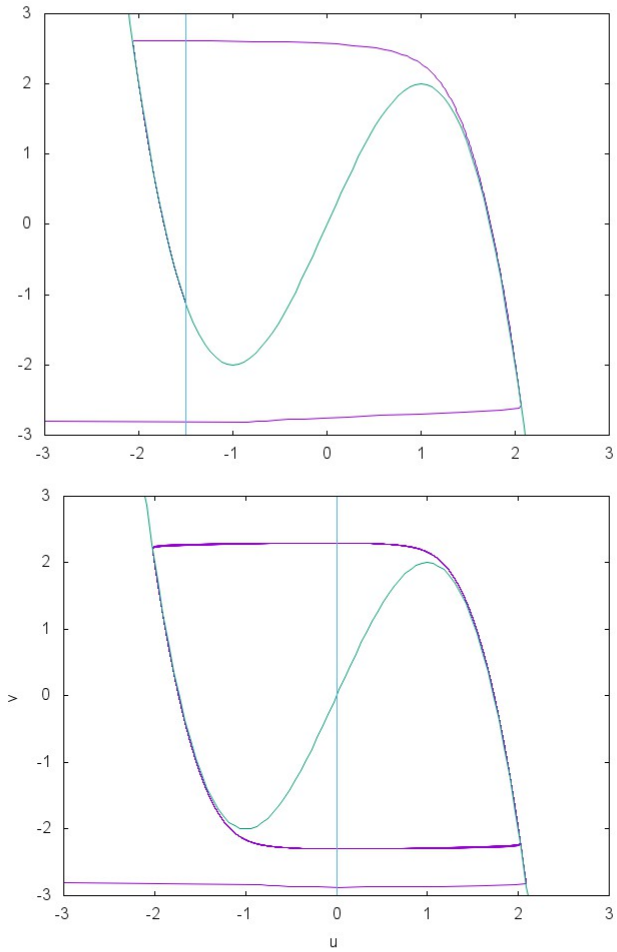

Figure 1: Solutions of system (3), for typical values of c.

As the c parameter is space dependent, we can couple oscillatory and excitable cells via the diffusion term. We put some center cells in an oscillatory regime and the others in an excitable state. We then address the question of wave propagation: will the center oscillations propagate along the excitable cells? We prove theoretically and show numerically that this depends on a parameter of excitability of the excitable cells. Varying this parameter, system (2) exhibits stable behavior, propagation of oscillations. This phenomenon occurs through an Hopf bifurcation in the infinite dimensional system (2). Note that we have already exploited such an idea in

[Krupa, Ambrosio & Alaoui] in the case of two coupled ODE slow-fast systems. The article is divided as follows: we study the spectrum properties of the linearized system of (2) in the second section. In the third part, we apply the center manifold theorem and compute restricted equations. Finally, in the fourth section we investigate numerically the phenomenon.

2 Hopf bifurcation for system (2)

As in the case of ODE's,the linear stability analysis near the stationary solution gives some insights on the qualitative behavior of the system. Some theories have been developed, see [Carr(1982), Henry(1981), Kuznetsov(1998)], however, the rigorous proofs in infinite systems are quite intricate and there is not a well established unified theory as in the finite case. In this short paper, we will concentrate on the spectral properties of the linearized operator. In this section we shall prove the existence of a Hopf bifurcation for system (2). Some linear stability analysis for reaction-diffusion FitzHugh-Nagumo (FHN) systems has already been studied, see for example [Chafee & Infantee(1974), Freitas & Rocha(2001), Rauch & Smoller(1978)]. The first article introducing a non-homogeneous term in the FHN Reaction Diffusion system is [Dikansky(2005)]. However, the following analysis involving such a non-homogeneous space dependent term $c(x)$, is new. We prove the positivity of an eigenvalue for small enough values of the bifurcation parameter by using classical spectral analysis. The remaining of the proof of the Hopf Bifurcation relies on techniques developed in chapter 5 of [Teschl(2010)]. After linearization near the stationary solution, we obtain an equation of regular Liouville type. Then, we introduce a polar change of coordinates. Then the result follows from comparison theorems for ODE's. We assume that the function $c(x)$, depending on a parameter $p > 0$, is regular and satisfies the following conditions:

$$c(x) \leq 0 \qquad \forall x \in [-a, a] \tag{4}$$

$$c(0) = 0 \tag{5}$$

$$c'(x) > 0 \quad \forall x \in]-a, 0[, c'(x) < 0 \forall x \in]0, a[\tag{6}$$

$$c'(-a) = c'(a) = 0 \tag{7}$$

$$\forall x \in [-a, a], x \neq 0, \quad c(x) \text{ is a decreasing function of } p \tag{8}$$

$$\text{and } \forall x \in [-a, a], x \neq 0, \qquad \lim_{p \to +\infty} c(x) = -\infty. \tag{9}$$

A typical function c if for example

$$c(x) = p(\frac{x^4}{a^4} - 2\frac{x^2}{a^2}).$$

Let $X = L^2([-a,a],\mathbb{R}^2)$, endowed with the scalar product,

$$< (u_1,v_1),(u_2,v_2) >= \int_{-a}^{a} u_1 u_2 dx + \int_{-a}^{a} v_1 v_2 dx$$

It is a classical question equation (2) generates a dynamical system on X. Now, let us remark that the stationary solution is given by:

$$\left\{ \begin{array}{rcl} \bar{v} & = & f(\bar{u}) + d_u \bar{u}_{xx} \\ \bar{u} & = & c(x) \end{array} \right. \tag{10}$$

The linearized system around $(\bar{u},\bar{v})$ is:

$$\left\{ \begin{array}{rcl} \epsilon u_t & = & f'(\bar{u})u - v + du_{xx} \\ v_t & = & u \end{array} \right. \tag{11}$$

We introduce the linear operator $\mathcal{F}$ with domain $\mathcal{D}(\mathcal{F})\{u,v \in H^2(]-a,a[); u'(-a) = u'(a) = 0\}$:

$$\mathcal{F}(u,v) = \left\{ \begin{array}{l} \frac{1}{\epsilon}\big(f'(\bar{u})u - v + du_{xx}\big) \\ u \end{array} \right.$$

We proceed to the spectral analysis. We look for functions u, v and numbers λ such that:

$$\left\{ \begin{array}{rl} \frac{1}{\epsilon}\big(f'(\bar{u})u - v + du_{xx}\big) & = \lambda u \\ u & = \lambda v \end{array} \right.$$

which is equivalent to

$$\left\{ \begin{array}{rl} f'(\bar{u})u - \frac{u}{\lambda} + du_{xx} & = \lambda\epsilon u \\ v & = \frac{u}{\lambda} \end{array} \right.$$

or,

$$\left\{ \begin{array}{rl} -du_{xx} - f'(\bar{u})u & = -\big(\frac{1}{\lambda} + \lambda\epsilon\big)u \\ v & = \frac{u}{\lambda} \end{array} \right.$$

We set:

$$\nu = -(\frac{1}{\lambda} + \lambda\epsilon),$$

then the first equation writes,

$$-du_{xx} - f'(\bar{u})u = \nu u, \tag{12}$$

and, we have,

$$\lambda_{-}^{+} = \frac{-\nu_{-}^{+}\sqrt{\nu^2 - 4\epsilon}}{2\epsilon}.$$

We have the classical following theorem,

Theorem 2. *There exists an increasing sequence of real numbers ν_n and an hilbertian basis $(u_n)_{n\in\mathbb{N}}$ of $L^2(\Omega)$ such that:*

$$\begin{array}{rcl} -du_{nxx} - f'(\bar{u})u_n & = & \nu_n u_n \\ u'(a) = u'(b) & = & 0. \end{array} \tag{13}$$

Furthermore,

$$\lim_{n\to+\infty} \nu_n = +\infty,$$

and,

$$\nu_0 = \inf_{u\in D(\mathcal{F});||u||=1} d\int_\Omega |\nabla u|^2 dx - \int_\Omega f'(\bar{u})u^2 dx. \tag{14}$$

We deduce the following proposition,

Proposition 1. *We assume that*

$$\int_\Omega f'(\bar{u})dx > 0 \tag{15}$$

then at least one eigenvalue of $\mathcal{F}$ has a positive real part.

Proof. We consider $u = \frac{1}{\sqrt{|\Omega|}}$. Then

$$\nu_0 \leq -\int_\Omega f'(\bar{u})u^2 dx < 0.$$

and

$$\lambda_{0}{}_{-}^{+} = \frac{-\nu_0{}_{-}^{+}\sqrt{\nu_0^2 - 4\epsilon}}{2\epsilon}$$

has a positive real part. □

Next, we prove that as p decreases from $+\infty$ to 0, the eigenvalue with the greatest real part crosses the imaginary axis from left to right, this proves the existence of the Hopf bifurcation.

Lemma 1. *The equation* (12)*:*

$$-du_{xx} - (\nu + f'(\bar{u}))u = 0,$$

rewrites

$$\theta_x = g(\theta) = \cos^2\theta + \frac{f'(\bar{u}) + \nu}{d}\sin^2\theta \tag{16}$$

$$r_x = \frac{\sin 2\theta}{2}(1 - \frac{\nu + f'(\bar{u})}{d})r \tag{17}$$

with

$$u = r\sin\theta, u_x = r\cos\theta. \tag{18}$$

Proof. We have

$$u_x = r_x\sin\theta + r\cos\theta\theta_x$$

and

$$u_{xx} = r_x\cos\theta - r\sin\theta\theta_x.$$

Multiplying the first equation by $\sin\theta$ and the second one by $\cos\theta$, adding both, we find:

$$r_x = r\frac{\sin(2\theta)}{2}(1 - \frac{\nu + f'(\bar{u})}{d}).$$

Multiplying the first equation by $\cos\theta$ and the second one by $-\sin\theta$, adding both, we find:

$$\theta_x = g(\theta) = \cos^2\theta + \frac{f'(\bar{u}) + \nu}{d}\sin^2\theta$$

or

$$\theta_x = g(\theta) = 1 + (\frac{f'(\bar{u}) + \nu}{d} - 1)\sin^2\theta.$$

□

The equation (16) depends only upon θ. Knowing θ, equation (17) gives:

$$r(x) = r(-a)\exp\int_{-a}^{x}\frac{\sin(2y)}{2}(1 - \frac{\nu + f'(\bar{u}(y))}{d})dy.$$

Therefore, we focus on the solutions of equation (16). As u verifies NBC, we restrict ourselves to solutions with $\theta(-a) = \frac{\pi}{2}$ and $\theta(a) = \frac{\pi}{2} \mod \pi$. We start with solutions of (16) vérifying $\theta(-a) = \frac{\pi}{2}$. Then we obtain a Cauchy problem.

By using comparison theorems, see for example [Teschl(2010)], we prove:

Proposition 2. *The function θ is positive over $]-a,a[$. For all $x \in]-a,a]$, $\theta(x)$ is an increasing function of ν and:*

$$\lim_{\nu \to -\infty} \theta(x) = 0,$$

$$\lim_{\nu \to +\infty} \theta(x) = +\infty,$$

Proof. Since,

$$\theta_x = g(\theta) = 1 + (\frac{f'(\bar{u}) + \nu}{d} - 1)\sin^2\theta$$

we have $\theta_x > 0$ for $\theta = 0$ which proves the first claim. Choose $\nu_2 > \nu_1$, then if θ_2 is a solution with ν_2 and θ_1 a solution with ν_1. Then $\theta_2 \geq \theta_1$. This follows by the comparison theorem, see appendix. Let $k > \frac{f'(\bar{u})+\nu}{d} - 1$. Then if $\bar{\theta}$ is a solution of,

$$\bar{\theta}_x = 1 + k\sin^2\bar{\theta} \tag{19}$$

then $\bar{\theta}$ is an upper solution of

$$\theta_x = g(\theta) = 1 + (\frac{f'(\bar{u}) + \nu}{d} - 1)\sin^2\theta$$

Now, for fixed x, and fixed $\gamma > 0$ small, there exists ν and $k > \frac{f'(\bar{u})+\nu}{d} - 1$ such that $\bar{\theta}(x) < \gamma$. As $\bar{\theta}(x)$ is an upper solution this shows our third claim. The last claim follows from same arguments.

□

Now, we will prove the following theorem,

Theorem 3. *For p small enough, the linearized operator $\mathcal{F}$ has at least one eigenvalue with positive real part. For p large enough, all the eigenvalues of the linearized system have negative real part. There is an Hopf Bifurcation: there exists a value p_0 for which as p crosses p_0 from right to left, the real part of a conjugate complex eigenvalues increases from negative to positive. The other eigenvalues remaining with negative real parts.*

Proof. We prove that:

1. for p large enough $\nu_0 > 0$, for p small enough $\nu_0 < 0$.
2. ν_0 is an increasing function of p.

We start with the first step.
For p small enough, since $f'(0) = 3$, $f'(\bar{u}(x)) > 0$ over $[-a, a]$ and the proposition 1 allows to conclude that $\nu_0 < 0$ in this case. Now, we deal with equation (16), with:

$$\theta(-a) = \frac{\pi}{2}, \, \nu = 0$$

Next, we prove that for p large enough:

$$\theta(a) < \frac{\pi}{2}.$$

Since θ is an increasing function of ν this implies that $\nu_0 > 0$. We will find an upper solution of equation (16) such that: $\theta(a) < \frac{\pi}{2}$.

Let w such that $w(-a) = \frac{\pi}{2}$ and

$$w(x) = \begin{cases} \frac{\pi}{2} - \alpha(x+a) & \text{if } x \in]-a, -\epsilon_1[\\ \frac{\pi}{2} - \alpha(-\epsilon_1 + a) + (1 + \frac{3}{d})(x + \epsilon_1) & \text{si } x \in]-\epsilon_1, \epsilon_1[\\ \frac{\pi}{2} - \alpha(-\epsilon_1 + a) + (1 + \frac{3}{d})2\epsilon_1 - \alpha(x - \epsilon_1) & \text{if } x \in]\epsilon_1, a[\end{cases}$$

This means that w is a continuous picewise linear function. Moreover, we choose α et ϵ_1 such that

$$\frac{\pi}{2} - 2a\alpha > 0,$$

which means

$$\alpha < \frac{\pi}{4a}.$$

This ensures that $w > 0$ over $[-a, a]$. Also, we choose

$$-\alpha(-\epsilon_1 + a) + (1 + \frac{3}{d})(2\epsilon_1) < 0,$$

which is equivalent to:

$$\alpha > (1 + \frac{3}{d})\frac{2\epsilon_1}{-\epsilon_1 + a}.$$

This is always possible ϵ_1 a soon as ϵ_1 small enough and ensures $w(x) < \frac{\pi}{2}$ over $]-a, a]$.

Then, in order to obtain a C^1 function, we slightly modify w, we set:

$$\begin{aligned} \tilde{w}(-a) &= w(-a) \\ \tilde{w}'(x) &= w'(x) \text{ on } [-a, -\epsilon_1[\cup [-\epsilon_2, \epsilon_2] \cup [\epsilon_1, a], \epsilon_2 < \epsilon_1 \\ \tilde{w}'(x) &= -\alpha + \tfrac{1+\frac{3}{d}+\alpha}{-\epsilon_2+\epsilon_1}(x + \epsilon_1) \text{ on } [-\epsilon_1, -\epsilon_2] \\ \tilde{w}'(x) &= -\alpha + \tfrac{1+\frac{3}{d}+\alpha}{\epsilon_2-\epsilon_1}(x - \epsilon_1) \text{ sur } [\epsilon_2, \epsilon_1] \end{aligned} \tag{20}$$

We rename $\tilde{w}, w$. Then for p large enough, we have:

$$w' > g(w).$$

For p large enough , $f'(\bar{u}(x)) < 0$ on $[-a,-\epsilon_2] \cup [\epsilon_2, a]$. Then for all $x \in [-a,-\epsilon_2[\cup[\epsilon_2, a]$:

$$g(w) < 1 + \frac{f'(\bar{u}(x))}{d} \inf_{x\in[-a,a]} \sin^2(w(x)).$$

Then, for p large enough,

$$\begin{aligned} g(w) < -\alpha \le w' \text{ over } [-a,-\epsilon_2[\cup[\epsilon_2, a], \\ g(w) \le 1 + \tfrac{3}{d} = w' \text{ over } [-\epsilon_2, \epsilon_2] \end{aligned} \tag{21}$$

This shows that w is an upper solution of (16), therefore $\theta < w$. It follows that, $\theta(a) < w(a) < \frac{\pi}{2}$. Therefore $\nu_0 > 0$ and all eigenvalues have a negative real part. Now, we prove that ν_0 is an increasing function of p. Since $\theta(a)$ is an increasing function of ν, it is sufficient to show that $\theta(a)$ is a decreasing function of p. Let $p_1 > p_2$ and let us denote by θ_1, g_1 (resp θ_2, g_2) the solution and the g function associated with p_1 (resp p_2), we have:

$$\dot{\theta}_1 - g_1(\theta_1) = 0$$

and

$$\begin{aligned} \dot{\theta}_2 - g_1(\theta_2) =& \dot{\theta}_2 - (\cos^2(\theta_2) + \frac{f'(\bar{u}_1) + \nu}{d} \sin^2(\theta_2)) \\ &= \dot{\theta}_2 - (\cos^2(\theta_2) + \frac{f'(\bar{u}_1) - f'(\bar{u}_2) + f'(\bar{u}_2) + \nu}{d} \sin^2(\theta_2)) \\ &= -(\frac{f'(\bar{u}_1) - f'(\bar{u}_2))}{d} \sin^2(\theta_2)) \\ &\ge 0. \end{aligned}$$

Therefore,

$$\dot{\theta}_1 - g_1(\theta_1) \le \dot{\theta}_2 - g_1(\theta_2).$$

Furthermore,

$$\dot{\theta}_1(-a) < \dot{\theta}_2(-a)$$

which implies that

$$\theta_2(x) > \theta_1(x) \text{ on }]-a, a],$$

which implies the result. □

3 Application of the center manifold theorem

In this section, we formally apply the procedure described in [Kuznetsov(1998)]. The theoretical analysis of the phenomenon using the framework of [Henry(1981)] is left for a forthcoming article. Let ϕ denote the dynamical system generated by equation (2) on X.

Theorem 4. *Let*

$$T^c = u_0(x)Vect\{(1,0),(0,1)\}.$$

There is a locally defined smooth two-dimensional invariant manifold $W^c \subset H$ that is tangent to T^c at 0. Moreover, there is a neighborhood U of $(\bar{u}, \bar{v}$, such that if $\phi(t)(u,v) \in U$ for all $t \geq 0$, then $\phi(t)(u,v) \to W^c$ for $t \to +\infty$. The equation on the manifold can be restricted to the complex equation

$$z_t = \lambda_1 z - \frac{3}{\epsilon}\int_\Omega \bar{u}u_0^3(z+\bar{z})^2 - \frac{6}{\epsilon}(z+\bar{z})\int_\Omega \bar{u}u_0^2 y_1 - \frac{3}{\epsilon}\int_\Omega u_0^4 z^2\bar{z} + ...$$

whereas the first lyapunov coefficient of the Hopf bifurcation is given by:

$$l_1(0) = -\frac{3}{2\sqrt{\epsilon}}(\int_\Omega u_0^4 + Re(\int_\Omega \bar{u}u_0^2 w_{20}^1))$$

with

$$-f'(\bar{u})w_{r20}^1 - (w_{r20}^1)_{xx} = -6\bar{u}u_0^2 + 12\epsilon\int_\Omega \bar{u}u_0^4$$

Proof. We define on the complexification of X the following scalar product:

$$((u_1,v_1),(u_2,v_2)) = \int_\Omega \bar{u}_1 u_2 + \int_\Omega \bar{v}_1 v_2$$

Then the adjoint operator $\mathcal{F}^t$ of $\mathcal{F}$ is given by:

$$\mathcal{F}^t(u,v) = \begin{cases} \frac{1}{\epsilon}\big(f'(\bar{u})u + v + d\Delta u\big) \\ -u \end{cases} \tag{22}$$

When $p = p_0$ the operator $\mathcal{F}$ has two purely complex conjugate eigenvalues λ_1 and λ_2, the others being of negative real part. We have,

$$\lambda_1 = \frac{i}{\sqrt{\epsilon}},\ \lambda_2 = -\frac{i}{\sqrt{\epsilon}}.$$

Let us denote by q the eigenvector associated to λ_1, then,

$$q(x) = u_0(x) \begin{pmatrix} 1 \\ -i\sqrt{\epsilon} \end{pmatrix}.$$

The eigenvalues of $\mathcal{F}^t$ are the same as those of $\mathcal{F}$. Let $\tilde{p}$ the eigenvector of $\mathcal{A}^t$ associated to λ_2. We find:

$$\tilde{p}(x) = q(x).$$

Furthermore,

$$(\tilde{p}, q) = 2\epsilon \int_\Omega u_0^2 dx.$$

Let

$$p = \frac{1}{2\int_\Omega u_0^2 dx} \tilde{p}.$$

Then:

$$(p, q) = 1.$$

Let

$$T^c = u_0(x) Vect\{(1,0),(0,1)\} = Vect\{re(q), im(q)\},$$

and

$$T^{su} = (T^c)^\perp.$$

Let $\xi = (u, v)$. We set:

$$\xi = zq + \bar{z}\bar{q} + y$$

with $y \in T^{su}$. Then $zq + \bar{z}\bar{q}$ is the orthogonal projection of ξ on T^c, and z, $\bar{z}$ are unique. We also verify that:

$$(p, \bar{q}) = 0 \text{ and } y \in T^{su} \Leftrightarrow (p, y) = 0.$$

It follows that:

$$\left\{ \begin{array}{lcl} z & = & (p, \xi) \\ y & = & \xi - (p, \xi)q - (\bar{p}, \xi)\bar{q} \end{array} \right.$$

Therefore, we obtain:

$$\left\{ \begin{array}{lcl} z_t & = & \lambda_1 z + (p, F(zq + \bar{z}\bar{q} + y)) \\ y_t & = & \mathcal{F}y + F(zq + \bar{z}\bar{q} + y) - (p, F(zq + \bar{z}\bar{q} + y))q - (\bar{p}, F(zq + \bar{z}\bar{q} + y))\bar{q} \end{array} \right.$$

with,

$$F(u, v) = \begin{pmatrix} -\frac{1}{\epsilon}(u^3 + 3\bar{u}u^2) \\ 0 \end{pmatrix}.$$

In our specific case, we obtain,

$$z_t = \lambda_1 z + \frac{1}{\epsilon}\int_\Omega u_0(-3\bar{u}((z+\bar{z})u_0+y_1)^2 - ((z+\bar{z})u_0+y_1)^3),$$

for the first equation and,

$$\begin{aligned} y_t =& \mathcal{F}y + \frac{1}{\epsilon}\begin{pmatrix} -3\bar{u}((z+\bar{z})u_0+y_1)^2 - ((z+\bar{z})u_0+y_1)^3 \\ 0 \end{pmatrix} \\ & - \int_\Omega \frac{u_0}{\epsilon}(-3\bar{u}((z+\bar{z})u_0+y_1)^2 - ((z+\bar{z})u_0+y_1)^3)\begin{pmatrix} 2\frac{u_0}{\int_\Omega u_0^2} \\ 0 \end{pmatrix} \end{aligned}$$

for the second one. In the first equation we only write the terms $(z+\bar{z})^2$ and $z^2\bar{z}$. We obtain,

$$z_t = \lambda_1 z - \frac{3}{\epsilon}\int_\Omega \bar{u}u_0^3(z+\bar{z})^2 - \frac{6}{\epsilon}(z+\bar{z})\int_\Omega \bar{u}u_0^2 y_1 - \frac{3}{\epsilon}\int_\Omega u_0^4 z^2\bar{z} + ... \quad (23)$$

In the second one, we only write the terms with order up to 2,

$$y_t = \mathcal{A}y - \frac{3}{\epsilon}(z+\bar{z})^2\begin{pmatrix} \bar{u}u_0^2 \\ 0 \end{pmatrix} + \frac{6}{\epsilon\int_\Omega u_0^2}(z+\bar{z})^2\int_\Omega \bar{u}u_0^3\begin{pmatrix} u_0 \\ 0 \end{pmatrix} + ... \quad (24)$$

It follows from the center manifold theorem that

$$y = \frac{w_{20}}{2}z^2 + w_{11}z\bar{z} + \frac{w_{02}}{2}\bar{z}^2 + O(|z|^3) \quad (25)$$

We derive (25) and identify with(24), we obtain:

$$\begin{cases} (2\lambda_1 Id - \mathcal{A})w_{20} &= H \\ -\mathcal{A})w_{11} &= H \\ (2\bar{\lambda}_1 Id - \mathcal{A})w_{02} &= H \end{cases}$$

with:

$$H = -\frac{6}{\epsilon}\begin{pmatrix} \bar{u}u_0^2 \\ 0 \end{pmatrix} + \frac{12}{\epsilon\int_\Omega u_0^2}\int_\Omega \bar{u}u_0^3\begin{pmatrix} u_0 \\ 0 \end{pmatrix}$$

This gives,

$$\begin{cases} (2\epsilon\lambda_1 - f'(\bar{u}) + \frac{1}{2\lambda_1})w_{20}^1 - d(w_{20}^1)_{xx} &= \epsilon H^1 \\ w_{20}^2 &= \frac{w_{20}^1}{2\lambda_1} \end{cases}$$

$$\begin{cases} w_{11}^1 &= 0 \\ w_{11}^2 &= \epsilon H^1 \end{cases}$$

$$\begin{cases} (2\epsilon\bar{\lambda}_1 - f'(\bar{u}) + \frac{1}{2\bar{\lambda}_1})w_{02}^1 - d(w_{02}^1)_{xx} & = & \epsilon H^1 \\ w_{02}^2 & = & \frac{w_{02}^1}{2\bar{\lambda}_1} \end{cases}$$

We rewrite in equation (23), we obtain:

$$z_t = \lambda_1 z - \frac{3}{\epsilon}\int_\Omega \bar{u}u_0^3(z+\bar{z})^2 + \big(-\frac{3}{\epsilon}\int_\Omega u_0^4 - \frac{6}{\epsilon}\int_\Omega \bar{u}u_0^2 w_{11}^1 - \frac{3}{\epsilon}\int_\Omega \bar{u}u_0^2 w_{20}^1\big)z^2\bar{z} + ... \tag{26}$$

The first Lyapunov coefficient of the Hopf bifurcation is given by:

$$l_1(0) = -\frac{3}{2\sqrt{\epsilon}}\left(\int_\Omega u_0^4 + Re\left(\int_\Omega \bar{u}u_0^2 w_{20}^1\right)\right)$$

□

4 Numerical simulations

For the numerical simulations, we choose $a = 1$ and

$$c(x) = p(x^4 - 2x^2)$$

Then we simulate equation (2) on $]-a, a[$ with an explcit scheme of Runge-Kutta 4 type, with a time step of $10^{-4}$and a space step of 0.1. The value of $\epsilon$is fixed at 0.1. We obtain:

- if $p > 2.1$ small enough, the solution converges towards a stationary solution. The figure 2 represents $u(x,t)$ for fixed $t = 550, 560, 570$ and $p = 2.1$. This solution do not change anymore and has reached the stationary solution. The figure 3 represents the solution $u(0,t)$ and $u(-1,t)$ for $t \in [500, 600]$.

- if $p < 2$, we observe periodic solutions. Figure 4 represents the solution $u(x,t)$ for fixed t large enough and $p = 2$. Figure 5 represents $u(0,t)$ and $u(-1,t)$ for $t \in [500, 600]$.

- Between these two values of p, there is a range of parameters for which we observe an intermediate behavior: the amplitude of the limit cycle decrease.

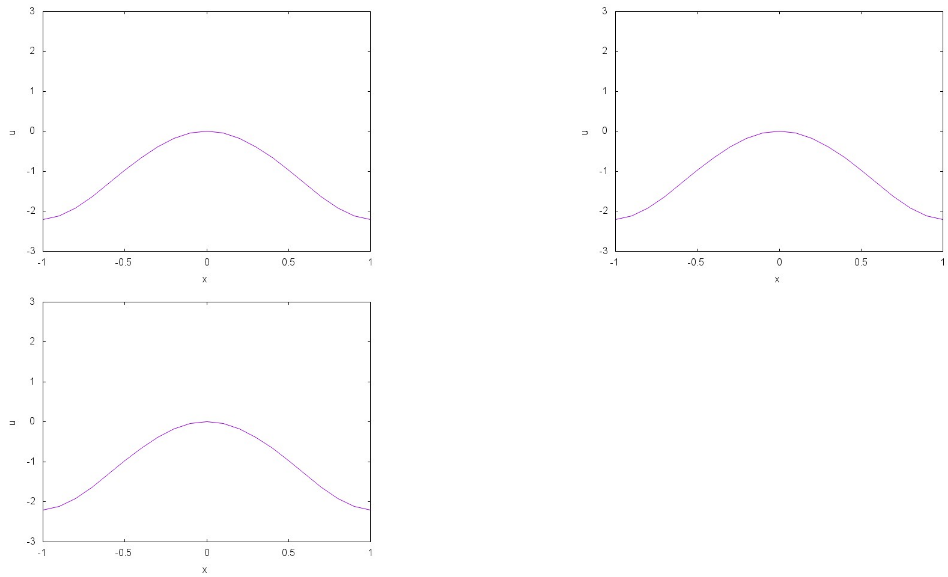

Figure 2: Bifurcation between stationary solution and periodic soltions. Stable stationary solution: u(x,t) for t=550, 560, 570 for $p = 2.1$.

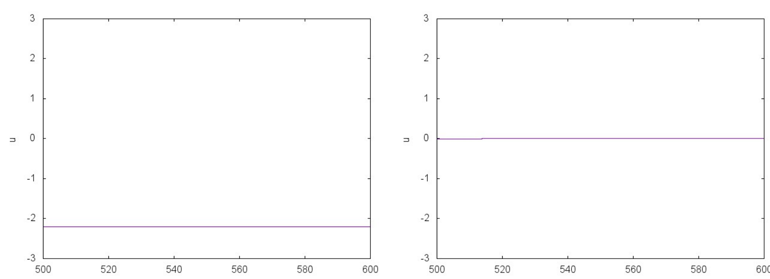

Figure 3: Bifurcation between stationary solution and periodic soltions. Stable stationary solution: $u(-1, t)$ and $u(0, t)$ for $t \in [500, 600]$ and $p = 2.1$.

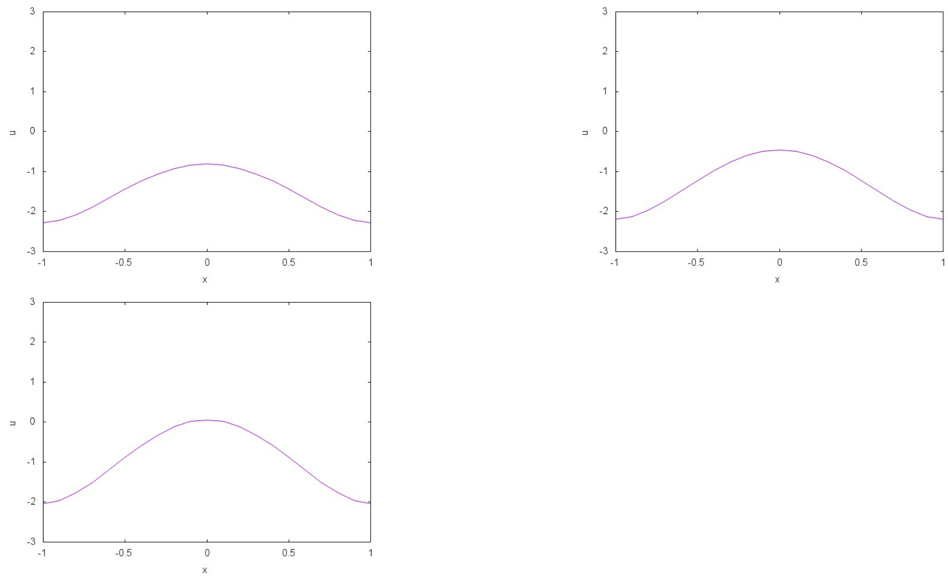

Figure 4: Bifurcation between stationary solution and periodic soltions. Stable periodic solution: u(x,t) for t=550, 560, 570 for $p = 2.0$.

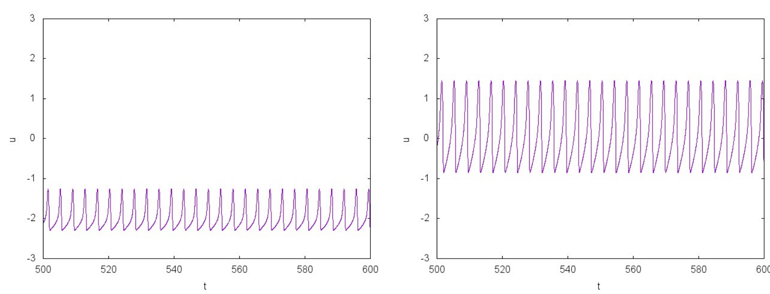

Figure 5: Bifurcation between stationary solution and periodic soltions. Stable stationary solution: $u(-1, t)$ and $u(0, t)$ for $t \in [500, 600]$ and $p = 2.0$.

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