Age-structured models with nonlocal diffusion: principal spectral theory, limiting properties and global dynamics

Hao Kang

Normandie Univ., UNIHAVRE, LMAH, France

March 4, 2021

joint work with Professor Shigui Ruan

Outline

1. **Introduction**
2. **Preliminaries**
3. **Principal Spectral Theory**
4. **Limiting Properties**
   - Without Kernel Scaling
   - With Kernel Scaling
5. **Strong Maximum Principle**
6. **Global Dynamics**

Structured Models

- In population dynamics, Metz and Diekmann 1986 studied the structured models
  The individual level $\overset{\text{structured models}}{\Longleftrightarrow}$ The population level

Structured Models

- In population dynamics, Metz and Diekmann 1986 studied the structured models
  The individual level $\overset{\text{structured models}}{\Longleftrightarrow}$ The population level
- In order to distinguish individuals, take their physiological conditions or physical characteristics such as age, size, location, status, and movement into consideration.

Structured Models

- In population dynamics, Metz and Diekmann 1986 studied the structured models
  The individual level $\overset{\text{structured models}}{\Longleftrightarrow}$ The population level
- In order to distinguish individuals, take their physiological conditions or physical characteristics such as age, size, location, status, and movement into consideration.
- The goal of structured population models is to understand how these physiological conditions or physical characteristics affect the dynamical properties of these models and thus the outcomes and consequences of the biological and epidemiological processes (Magal and Ruan 2018).

Age-structured Models with Spatial Diffusion

In recent years age-structured models with random diffusion have been used to model population dynamics with spatial internal intersections from the population and individual levels, see Gurtin, MacCamy, Langlais, Chan, Guo, Walker, Webb, Delgado and so on. Among some of them, the principal eigenvalues of age-structured models with random diffusion serve as a crucial tool for the investigation such equations, for example Chan and Guo, 1990 MM, JMAA 1994 and Delgado et. al, 2006 JMAA, 2008 JDE.

Age-structured Models with Spatial Diffusion

In recent years age-structured models with random diffusion have been used to model population dynamics with spatial internal intersections from the population and individual levels, see Gurtin, MacCamy, Langlais, Chan, Guo, Walker, Webb, Delgado and so on. Among some of them, the principal eigenvalues of age-structured models with random diffusion serve as a crucial tool for the investigation such equations, for example Chan and Guo, 1990 MM, JMAA 1994 and Delgado et. al, 2006 JMAA, 2008 JDE.

However, studies cooperating age-structure and nonlocal diffusion, i.e. age-structured models with nonlocal diffusion, are few to our best knowledge besides our work. In this paper we are continuing to devote to studying the criterion on the existence of principal eigenvalue, asymptotic behaviors of generalized principal eigenvalue with respect to diffusion rate and global dynamics of age-structured with nonlocal diffusion and KPP type of nonlinearity.

Eigenvalue Problem

$$\begin{cases} \frac{\partial u(a,x)}{\partial a} = \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)u(a,y)dy - u(a,x)\right] - \mu(a,x)u(a,x) \\ \qquad -\lambda u(a,x), a>0, x\in\overline{\Omega}, \\ u(0,x) = \int_0^{a^+}\beta(a,x)u(a,x)da, \qquad x\in\overline{\Omega}. \end{cases} \tag{1}$$

where $a^+ < \infty$ represents the maximum age and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, diffusion rate $D > 0$, diffusion range $\sigma > 0$ and cost parameter $m \in [0, 2)$, with $J_\sigma(x) = \frac{1}{\sigma^N} J\left(\frac{x}{\sigma}\right)$ for $x \in \mathbb{R}^N$. The diffusion kernel $J \in C(\mathbb{R}^N)$ is the nonnegative and supported in $B(0, r)$ for some $r > 0$, and satisfies $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$. Assume that birth rate $\beta(a, x)$ and death rate $\mu(a, x)$ are positive and belong to $C^{0,1}([0, a^+] \times \overline{\Omega})$ and define

$$\underline{\mu}(a) := \inf_{x\in\overline{\Omega}} \mu(a,x), \quad \overline{\mu}(a) := \sup_{x\in\overline{\Omega}} \mu(a,x),$$

$$\underline{\beta}(a) := \inf_{x\in\overline{\Omega}} \beta(a,x), \quad \overline{\beta}(a) := \sup_{x\in\overline{\Omega}} \beta(a,x),$$

A Few Remarks

(i) Here we assumed the domain $\Omega \subset \mathbb{R}^N$ is bounded and the kernel J is compactly supported simultaneously. In fact, for the existence of principal eigenvalue it is only needed to require Ω to be bounded. And the assumption that J has compact support is only need to investigate the limiting properties, see Berestycki et al, 2016 JFA.

(ii) If one only studies the generalized principal eigenvalue with their properties, the assumption Ω is bounded can also be removed, see Berestycki et al, 2016 JFA.

(iii) Here in order to give a comprehensive result, we provided a unified assumption.

Age-Structured Models

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) = -\mu(a)u(t,a), \quad t, a > 0, \\ u(t,0) = \int_0^{a^+} \beta(a)u(t,a)da, \quad t \geq 0, \\ u(0,a) = u_0(a) \in L^1(0,a^+) \end{cases} \tag{2}$$

Age-Structured Models

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) = -\mu(a)u(t,a), \quad t, a > 0, \\ u(t,0) = \int_0^{a^+} \beta(a)u(t,a)da, \quad t \geq 0, \\ u(0,a) = u_0(a) \in L^1(0,a^+) \end{cases} \tag{2}$$

By the method of characteristic lines, we can solve the first equation to obtain

$$u(t,a) = \begin{cases} \Pi(a, a-t)u_0(a-t), \quad 0 \leq t \leq a < a^+, \\ \Pi(a,0)u(t-a,0), \quad t > a, 0 \leq a < a^+ \end{cases} \tag{3}$$

where $\Pi(a,\sigma) := e^{-\int_\sigma^a \mu(\tau)d\tau}, 0 \leq \sigma < a$ can be interpreted as the probability that an individual of age σ survives to age a.

Age-Structured Models

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) = -\mu(a)u(t,a), \quad t,a>0, \\ u(t,0) = \int_0^{a^+} \beta(a)u(t,a)da, \quad t \geq 0, \\ u(0,a) = u_0(a) \in L^1(0,a^+) \end{cases} \tag{2}$$

By the method of characteristic lines, we can solve the first equation to obtain

$$u(t,a) = \begin{cases} \Pi(a,a-t)u_0(a-t), \quad 0 \leq t \leq a < a^+, \\ \Pi(a,0)u(t-a,0), \quad t > a, 0 \leq a < a^+ \end{cases} \tag{3}$$

where $\Pi(a,\sigma) := e^{-\int_\sigma^a \mu(\tau)d\tau}, 0 \leq \sigma < a$ can be interpreted as the probability that an individual of age σ survives to age a.
Thus $U(t) := u(t,0)$ satisfies the Volterra equation

$$U(t) = \int_0^t h(a)\beta(a)\Pi(a,0)U(t-a)da + \int_t^{a^+} h(a)\beta(a)\Pi(a,a-t)u_0(a-t)da$$

for $t \geq 0$ with cutoff function $h(a) := 1$ if $a \in (0,a^+)$ and $h(a) := 0$ otherwise, which is also known as *renewal equation*.

Nonlocal Diffusion

In the meanwhile there is an increasing interest in nonlocal diffusion problems modeled by convolution diffusion operators such as

$$Lv := d \int_{\Omega} J(x-y)[v(y)-v(x)]dy,$$

where $v \in X$ and X is a proper Banach space. J is the diffusion kernel which is a C^0 and nonnegative function with unit integral representing the spatial dispersal, i.e.,

$$\int_{\mathbb{R}^N} J(x)dx = 1, \quad J(x) \geq 0, \quad \forall x \in \mathbb{R}^N, \quad J(0) > 0.$$

Nonlocal Diffusion

In the meanwhile there is an increasing interest in nonlocal diffusion problems modeled by convolution diffusion operators such as

$$Lv := d \int_{\Omega} J(x-y)[v(y)-v(x)]dy,$$

where $v \in X$ and X is a proper Banach space. J is the diffusion kernel which is a C^0 and nonnegative function with unit integral representing the spatial dispersal, i.e.,

$$\int_{\mathbb{R}^N} J(x)dx = 1, \quad J(x) \geq 0, \quad \forall x \in \mathbb{R}^N, \quad J(0) > 0.$$

The convolution $J(x-y)$ is viewed as the probability distribution of jumping from location y to location x, namely the convolution $\int_{\Omega} J(x-y)v(y)dy$ is the rate at which individuals are arriving to position x from other places and $\int_{\Omega} J(y-x)v(x)dy$ is the rate at which they are leaving location x to travel to other sites.

Commons and Differences

Laplace and nonlocal diffusions share many properties for instance, both of them have maximum principle, bounded stationary solutions are constants.

Commons and Differences

Laplace and nonlocal diffusions share many properties for instance, both of them have maximum principle, bounded stationary solutions are constants. However,

(i) The solution operator for nonlocal diffusion corresponding to this initial value problem is not a smoothing operator;

(ii) The semiflows generated by nonlocal diffusion is not compact with respect to the compact open topology;

(iii) The solutions for nonlocal diffusion usually loss the spatial regularity;

(iv) The operator of nonlocal diffusion may not have principal eigenvalues;

(v) The spatial decay rates of traveling waves at infinity differ in the two cases.

Principal Eigenvalue of Age-Structure Operator

$$[B\eta](a) = -\frac{\partial \eta(a)}{\partial a} - \mu(a)\eta(a), \quad \forall \eta \in D(B), \tag{4}$$

$$D(B) = \{\eta(a)|\eta, B\eta \in L^1(0, a^+), \eta(0) = \int_0^{a^+} \beta(a)\eta(a)da\}$$

$\{\gamma_j\}_{j\geq 0}$ are the eigenvalues of B, i.e., the solution of the following equation

$$F(\gamma) := \int_0^{a^+} \beta(a)e^{-\gamma a}\pi(a)da = 1,$$

where $\pi(a) = e^{-\int_0^a \mu(\rho)d\rho}$. It follows that $\gamma \in \sigma(B) \Longleftrightarrow F(\gamma) = 1$. B has the unique real eigenvalue γ_0 with algebraic multiplicity 1. Moreover, γ_0 is the principal eigenvalue of B, that is,

$$\gamma_0 > \mathrm{Re}\gamma_1 \geq \mathrm{Re}\gamma_2 \geq \cdots$$

Principal Eigenvalue for Nonlocal Operators

(i) Coville, 2010 JDE, studied the principal eigenvalue via generalized principal eigenvalue (Berestycki et. al, 1994 CPAM) and gave a non-locally-integrable condition based on the generalized Krein Rutman theorem (Edmunds et. al, 1972 PRSLA).

Principal Eigenvalue for Nonlocal Operators

(i) Coville, 2010 JDE, studied the principal eigenvalue via generalized principal eigenvalue (Berestycki et. al, 1994 CPAM) and gave a non-locally-integrable condition based on the generalized Krein Rutman theorem (Edmunds et. al, 1972 PRSLA).

(ii) Berestycki et. al, 2016 JFA, investigated the asymptotic behavior of generalized principal eigenvalue on the diffusion rate.

Principal Eigenvalue for Nonlocal Operators

(i) Coville, 2010 JDE, studied the principal eigenvalue via generalized principal eigenvalue (Berestycki et. al, 1994 CPAM) and gave a non-locally-integrable condition based on the generalized Krein Rutman theorem (Edmunds et. al, 1972 PRSLA).

(ii) Berestycki et. al, 2016 JFA, investigated the asymptotic behavior of generalized principal eigenvalue on the diffusion rate.

(iii) Rawal and Shen, 2012 JDDE, Shen and Xie 2015 DCDS, studied the existence of principal eigenvalue for autonomous and time-periodic cases using the idea of perturbation of positive operators from Burger, 1988 MZ.

Principal Eigenvalue for Nonlocal Operators

(i) Coville, 2010 JDE, studied the principal eigenvalue via generalized principal eigenvalue (Berestycki et. al, 1994 CPAM) and gave a non-locally-integrable condition based on the generalized Krein Rutman theorem (Edmunds et. al, 1972 PRSLA).

(ii) Berestycki et. al, 2016 JFA, investigated the asymptotic behavior of generalized principal eigenvalue on the diffusion rate.

(iii) Rawal and Shen, 2012 JDDE, Shen and Xie 2015 DCDS, studied the existence of principal eigenvalue for autonomous and time-periodic cases using the idea of perturbation of positive operators from Burger, 1988 MZ.

(iv) Shen and Vo, 2019 JDE and Su et. al, 2020 JDE studied the asymptotic behavior of the generalized principal eigenvalue on the diffusion rate under the time-periodic case.

Principal Eigenvalue of Nonlocal Diffusion Operator (Dirichlet)

(i) Autonomous (Shen & Xie, DCDS 2015):

$$[L\phi](x) := D\left[\int_\Omega J(x-y)\phi(y)dy - \phi(x)\right] + a(x)\phi(x), \quad \phi \in C(\overline{\Omega}).$$

The principal eigenvalue of L exists iff $\lambda_1(L) > h_{\max}$, where $\lambda_1(L) = s(L), h_{\max} := \max_{x\in\overline{\Omega}}(-D + a(x))$. Moreover,

$$\lim_{D\to 0^+} \lambda_1(L) = \max_{x\in\overline{\Omega}} a(x), \quad \lim_{D\to\infty} \lambda_1(L) = -\infty.$$

Principal Eigenvalue of Nonlocal Diffusion Operator (Dirichlet)

(i) Autonomous (Shen & Xie, DCDS 2015):

$$[L\phi](x) := D\left[\int_\Omega J(x-y)\phi(y)dy - \phi(x)\right] + a(x)\phi(x), \quad \phi \in C(\overline{\Omega}).$$

The principal eigenvalue of L exists iff $\lambda_1(L) > h_{\max}$, where $\lambda_1(L) = s(L)$, $h_{\max} := \max_{x\in\overline{\Omega}}(-D + a(x))$. Moreover,

$$\lim_{D\to 0^+} \lambda_1(L) = \max_{x\in\overline{\Omega}} a(x), \quad \lim_{D\to\infty} \lambda_1(L) = -\infty.$$

(ii) Time-Periodic (Rawal & Shen, JDDE 2012, Shen & Vo, JDE 2019):

$$[\tilde{L}\phi](t,x) := -\partial_t\phi(t,x) + D\left[\int_\Omega J(x-y)\phi(t,y)dy - \phi(t,x)\right] + a(t,x)\phi(t,x),$$

for $\phi \in C_T(\mathbb{R}\times\overline{\Omega})$. The principal eigenvalue of $\tilde{L}$ exists iff $\lambda_1(\tilde{L}) > \tilde{h}_{\max}$, where $\lambda_1(\tilde{L}) = s(\tilde{L})$, $\tilde{h}_{\max} := \max_{x\in\overline{\Omega}}(-D + \hat{a}(x))$ and $\hat{a} := \frac{1}{T}\int_0^T a(t,x)dt$. Moreover,

$$\lim_{D\to 0^+} \lambda_1(\tilde{L}) = \max_{x\in\overline{\Omega}} \hat{a}(x), \quad \lim_{D\to\infty} \lambda_1(\tilde{L}) = -\infty.$$

Our age-structured models with nonlocal diffusion

(i) We can not directly choose the space of functions satisfying the integral condition as in Rawal and Shen, 2012 JDDE, where they worked on the space of time periodic functions, since such a function space is unknown and heavily depends on the birth rate β.

Our age-structured models with nonlocal diffusion

(i) Thus it forces us to define the operator $\mathcal{A}$ given in (5) containing the integral boundary condition.

Our age-structured models with nonlocal diffusion

(i) Thus it forces us to define the operator $\mathcal{A}$ given in (5) containing the integral boundary condition.

(ii) We introduce the theory of resolvent positive operators with their perturbations that were studied by Thieme 1998 DCDS and 2009 SIAM JAM, and Kato, 1982 MZ which is basically like Burger's idea, 1988 MZ to provide a criterion on the existence of the principal eigenvalue.

Our age-structured models with nonlocal diffusion

(i) Thus it forces us to define the operator $\mathcal{A}$ given in (5) containing the integral boundary condition.

(ii) We introduce the theory of resolvent positive operators with their perturbations that were studied by Thieme 1998 DCDS and 2009 SIAM JAM, and Kato, 1982 MZ which is basically like Burger's idea, 1988 MZ to provide a criterion on the existence of the principal eigenvalue.

(iii) Observing our case, since it contains the ∂_a term, which is like a parabolic type of nonlocal operator, it does not admit the usual L^2 variational structure in the elliptic type cases. This fact suggests us to follow Berestycki's idea to study the generalized principal eigenvalue when investigating their asymptotic behavior with respect to diffusion rate.

Function Spaces and Operators

$$\begin{aligned}
\mathcal{X} &= X \times L^1((0,a^+),X), \quad \mathcal{X}_0 = \{0\} \times L^1((0,a^+),X), \quad X := C(\overline{\Omega}),\\
\mathcal{X}_0^+ &= \{0\} \times L^1_+((0,a^+),X)\\
&= \{0\} \times \{u \in L^1((0,a^+),X) : u(a,x) \geq 0, (a,x) \in (0,a^+) \times \overline{\Omega}\},\\
\mathcal{X}_0^{++} &= \{0\} \times L^1_{++}((0,a^+),X)\\
&= \{0\} \times \{u \in L^1((0,a^+),X) : u(a,x) > 0, (a,x) \in (0,a^+) \times \overline{\Omega}\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}(0,f) &= \left(-f(0,\cdot), \quad -f' + L_{\sigma,m,\Omega} f\right),\\
\mathcal{C}(0,f) &= \left(\int_0^{a^+} \beta(a,\cdot) f(a,\cdot) da, \quad 0\right), (0,f) \in \mathcal{X}_0.
\end{aligned}$$

where

$$L_{\sigma,m,\Omega} f = \frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y) f(a,y) dy - f(a,x)\right] - \quad \mu(a,x) f(a,x),$$

$$f \in L^1((0,a^+),X)$$

Continued

Note that $\mathcal{X}_0$ is a Banach space with a positive cone $\mathcal{X}_0^+$ which is normal and generating. $\mathcal{X}_0$ can be identified with $L^1((0,a^+),X)$ in an obvious way. Let

$$\mathcal{A}=\mathcal{B}+\mathcal{C}, \quad \text{with domain} \quad D(\mathcal{A})=\{0\}\times W^{1,1}((0,a^+),X)^1, \quad (5)$$

where $W^{1,1}$ represents the weak differentiability in a. Define $\mathcal{A}_0$ be the part of $\mathcal{A}$ in $\mathcal{X}_0$,

$$D(\mathcal{A}_0)=\{(0,f);\mathcal{A}f\in\mathcal{X}_0\}.$$

Then $(0,f)\in D(\mathcal{A}_0)$ implies that $f(0,\cdot)=\int_0^{a^+}\beta(a,\cdot)f(a,\cdot)da$, i.e. the boundary condition in (1). Moreover, define the nonlocal operator for $f\in L^1((0,a^+),X)$ as

$$L^0_{\sigma,m,\Omega}[f](a,x)=\frac{D}{\sigma^m}\left[\int_\Omega J_\sigma(x-y)f(a,y)dy-f(a,x)\right]. \quad (6)$$

[1] $W^{1,1}((0,a^+),X)\hookrightarrow C([0,a^+],X)$

Resolvent Positive Operators

Definition 1

A closed operator A in Z is called *resolvent positive* if the resolvent set of A, $\rho(A)$, contains a ray (ω, ∞) and $(\lambda - A)^{-1}$ is a positive operator (i.e. maps Z_+ into Z_+) for all $\lambda > \omega$.

Let $A = B + C$, where B is resolvent positive and C is linear positive, define $F_\lambda := C(\lambda - B)^{-1}, \lambda > s(B)$. Then

- **a** $r(F_\lambda) \geq 1$ for all $\lambda > s(B)$, then A is not resolvent positive;
- **b** $r(F_\lambda) < 1$ for all $\lambda > s(B)$, then A is resolvent positive and $s(A) = s(B)$;
- **c** There exists $\nu > \lambda > s(B)$ such that $r(F_\nu) < 1 \leq r(F_\lambda)$: then A is resolvent-positive and $s(B) < s(A) < \infty$; further $s = s(A)$ is characterized by $r(F_s) = 1$.

Principal Spectral Theory

$\mathcal{A} = \mathcal{B} + \mathcal{C}$ with for $(0, f) \in D(\mathcal{A})$,

$$\mathcal{B}(0, f) = \left(-f(0, \cdot), \quad -f' + Lf\right), \mathcal{C}(0, f) = \left(\int_0^{a^+} \beta(a, \cdot) f(a, \cdot) da, \quad 0\right).$$

where for $f \in L^1((0, a^+), X)$,

$$L[f](a, x) = D\left[\int_\Omega J(x - y) f(a, y) dy - f(a, x)\right] - \mu(a, x) f(a, x),$$

Principal Spectral Theory

$\mathcal{A} = \mathcal{B} + \mathcal{C}$ with for $(0, f) \in D(\mathcal{A})$,

$$\mathcal{B}(0,f) = \left(-f(0,\cdot), \quad -f' + Lf\right), \mathcal{C}(0,f) = \left(\int_0^{a^+} \beta(a,\cdot)f(a,\cdot)da, \quad 0\right).$$

where for $f \in L^1((0,a^+),X)$,

$$L[f](a,x) = D\left[\int_\Omega J(x-y)f(a,y)dy - f(a,x)\right] - \mu(a,x)f(a,x),$$

Definition 2

The *principal spectrum point* of $\mathcal{A}$ is defined by $\lambda_1(\mathcal{A}) = \sup\{\Re\lambda : \lambda \in \sigma(\mathcal{A})\}$. If $\lambda_1(\mathcal{A})$ is an isolated eigenvalue of $\mathcal{A}$ with an eigenfunction in $\mathcal{X}_0^{++} \cap D(\mathcal{A})$, then it is called *principal eigenvalue* of $\mathcal{A}$.

Note that $\lambda_1(\mathcal{A}) = s(\mathcal{A})$, which is the spectral bound of $\mathcal{A}$.

Decomposition

Define

$$\mathcal{B}_1(0,f) = \left(-f(0,\cdot), \quad -f' - (D+\mu)f\right),$$

$$\mathcal{B}_2(0,f) = \left(0, \quad D\int_\Omega J(\cdot - y)f(a,y)dy\right), \quad (0,f) \in D(\mathcal{A}).$$

It's obvious that $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$.

Decomposition

Define

$$\mathcal{B}_1(0,f) = \left(-f(0,\cdot), \quad -f' - (D+\mu)f\right),$$
$$\mathcal{B}_2(0,f) = \left(0, \quad D\int_\Omega J(\cdot - y)f(a,y)dy\right), \quad (0,f) \in D(\mathcal{A}).$$

It's obvious that $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$.
Observe that if $\alpha \in \mathbb{C}$ is such that $(\alpha - \mathcal{B}_1 - \mathcal{C})^{-1}$ exists, then

$$(\mathcal{B}_2 + \mathcal{B}_1 + \mathcal{C})u = \alpha u$$

has nontrivial solutions in $\mathcal{X}_0 \oplus i\mathcal{X}_0$ is equivalent to

$$\mathcal{B}_2(\alpha - \mathcal{B}_1 - \mathcal{C})^{-1}v = v$$

has nontrivial solutions in $\mathcal{X} \oplus i\mathcal{X}$, where

$$\mathcal{X}_0 \oplus i\mathcal{X}_0 = \{u + iv | u, v \in \mathcal{X}_0\}, \quad \mathcal{X} \oplus i\mathcal{X} = \{u + iv | u, v \in \mathcal{X}\}.$$

A Key Proposition

Proposition 1

$(\alpha - \mathcal{B}_1 - \mathcal{C})^{-1}$ exists when $\Re\alpha > \alpha^{**}$, where $\alpha^{**} \in \mathbb{R}$ satisfying

$$r(\mathcal{G}_{\alpha^{**}}) = r\left(\int_0^{a^+} \beta(a,x)e^{-(\alpha^{**}+D)a}\Pi(0,a,x)da\right) = 1, \tag{7}$$

where $\Pi(\gamma, a, x) := e^{-\int_\gamma^a \mu(s,x)ds}$, $\mathcal{G}_\alpha : X \to X$ is a linear bounded operator defined in the following,

$$[\mathcal{G}_\alpha g](x) = \int_0^{a^+} \beta(a,x)e^{-(\alpha+D)a}\Pi(0,a,x)g(x)da, \quad g \in X. \tag{8}$$

Moreover, $\mathcal{B}_1 + \mathcal{C}$ is a resolvent positive operator. In addition, $s(\mathcal{B}_1 + \mathcal{C}) = \alpha^{**}$ and α^{**} also satisfies the following equation,

$$\max_{x \in \overline{\Omega}} \int_0^{a^+} \beta(a,x)e^{-(\alpha^{**}+D)a}\Pi(0,a,x)da = 1. \tag{9}$$

With Nonlocal Diffusion

$$\begin{cases} \frac{\partial u(a,x)}{\partial a} = D\int_\Omega J(x-y)u(a,y)dy - Du(a,x) - & \mu(a,x)u(a,x), \\ & (a,x) \in (0,a^+) \times \overline{\Omega} \\ u(\tau,x) = \phi(x) \in X, \end{cases}$$

Define the evolution system $\{\mathcal{U}(\tau,a)\}_{0\le\tau\le a\le a^+}$ associated with above equation, that is the solution $u(a,x)$ of (10) can be written as

$$u(a,x) = \mathcal{U}(\tau,a)\phi(x). \tag{10}$$

With Nonlocal Diffusion

$$\begin{cases} \frac{\partial u(a,x)}{\partial a} = D\int_\Omega J(x-y)u(a,y)dy - Du(a,x) - & \mu(a,x)u(a,x), \\ & (a,x) \in (0,a^+) \times \overline{\Omega} \\ u(\tau,x) = \phi(x) \in X, \end{cases}$$

Define the evolution system $\{\mathcal{U}(\tau,a)\}_{0\le\tau\le a\le a^+}$ associated with above equation, that is the solution $u(a,x)$ of (10) can be written as

$$u(a,x) = \mathcal{U}(\tau,a)\phi(x). \tag{10}$$

Proposition 2

$\mathcal{A}$ is resolvent positive and $s(\mathcal{A}) = \lambda_0$ where λ_0 satisfies

$$r(\mathcal{M}_{\lambda_0}) = r\left(\int_0^{a^+} \beta(a,x)e^{-\lambda_0 a}\mathcal{U}(0,a)da\right) = 1, \tag{11}$$

where $\mathcal{M}_\lambda : X \to X$ is a linear bounded operator defined in above.

Observation

$\mathcal{A}$ is resolvent positive implies $s(\mathcal{A}) \geq s(\mathcal{B}_1 + \mathcal{C})$.

Remark 3

But we can not obtain the strict relation, i.e. $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ even if

$$e^{-Da}\Pi(0, a, x) \ll \mathcal{U}(0, a)$$

holds, because α^{**} and λ_0 are obtained by taking the spectral radius of the operators equal to 1 where a limit process occurs in which the strict relation may not be preserved. However, if $r(\mathcal{G}_\alpha)$ and $r(\mathcal{M}_\lambda)$ are eigenvalues of $\mathcal{G}_\alpha$ and $\mathcal{M}_\lambda$ respectively, we could obtain the strict relation, by the Frobenius theory for positive operators.

Observation

$\mathcal{A}$ is resolvent positive implies $s(\mathcal{A}) \geq s(\mathcal{B}_1 + \mathcal{C})$.

Remark 3

But we can not obtain the strict relation, i.e. $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ even if

$$e^{-Da}\Pi(0,a,x) \ll \mathcal{U}(0,a)$$

holds, because α^{**} and λ_0 are obtained by taking the spectral radius of the operators equal to 1 where a limit process occurs in which the strict relation may not be preserved. However, if $r(\mathcal{G}_\alpha)$ and $r(\mathcal{M}_\lambda)$ are eigenvalues of $\mathcal{G}_\alpha$ and $\mathcal{M}_\lambda$ respectively, we could obtain the strict relation, by the Frobenius theory for positive operators.

Proposition 3

$s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ if $\mu(a,x) \equiv \mu(a)$ and $\beta(a,x) \equiv \beta(a)$;

A Criterion of Existence of Principal Eigenvalue

Now we provide a sufficient condition to make the principal spectrum point $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ become the principal eigenvalue.

Theorem 4 (K.-Ruan, 2020)

If $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$, then $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. Conversely, if $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{A}$ with a positive eigenfunction $\phi(a, x)$, then $\lambda = s(\mathcal{A})$.

Denote

$$\mathcal{F}_\lambda = \mathcal{B}_2(\lambda - \mathcal{B}_1 - \mathcal{C})^{-1}, \quad \Re\lambda > \alpha^{**}. \tag{12}$$

A Criterion of Existence of Principal Eigenvalue

Now we provide a sufficient condition to make the principal spectrum point $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ become the principal eigenvalue.

Theorem 4 (K.-Ruan, 2020)

If $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$, then $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. Conversely, if $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{A}$ with a positive eigenfunction $\phi(a, x)$, then $\lambda = s(\mathcal{A})$.

Denote

$$\mathcal{F}_\lambda = \mathcal{B}_2(\lambda - \mathcal{B}_1 - \mathcal{C})^{-1}, \quad \Re\lambda > \alpha^{**}. \tag{12}$$

Corollary 5

The inequality $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ holds if and only if there is $\lambda^* > s(\mathcal{B}_1 + \mathcal{C})$ such that $r(\mathcal{F}_{\lambda^*}) \geq 1$, where $\mathcal{F}_\lambda$ is defined in (12).

Main Theorem

Theorem 6 (K.-Ruan, 2020)

If for every $\alpha > \alpha^{**}$,

$$\frac{1}{1-\mathcal{G}_\alpha} \notin L^1_{loc}(\overline{\Omega}), \tag{13}$$

then $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ *is the principal eigenvalue of* $\mathcal{A}$*, where* $\mathcal{G}_\alpha(x)$ *is defined in the following,*

$$\mathcal{G}_\alpha(x) = \int_0^{a^+} \beta(a,x) e^{-(\alpha+D)a} \Pi(0,a,x) da, \tag{14}$$

Moreover, (13) is equivalent to for every $\zeta > \alpha^{**}$,

$$\frac{1}{\zeta - \alpha} \notin L^1_{loc}(\overline{\Omega}).$$

A Counter Example (implies (13) sharp in some sense)

$$\begin{cases} \frac{\partial \phi(a,x)}{\partial a} = \int_\Omega J(x-y)\phi(a,y)dy - \phi(a,x) - \quad \mu\phi(a,x) - \lambda_1\phi(a,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a,x) \in (0,a^+) \times \overline{\Omega}, \\ \phi(0,x) = \int_0^\infty \beta(x)\phi(a,x)da, \qquad\qquad\qquad x \in \overline{\Omega}. \end{cases}$$

Now for any $\alpha > \alpha_{\max}$, where $\alpha_{\max} \in \mathbb{R}$ such that

$$\int_0^\infty e^{-(\alpha_{\max}+1+\mu)a} da = 1/\beta_{\max},$$

where $\beta_{\max} := \max_{x \in \overline{\Omega}} \beta(x)$. It follows that

$$\rho/\beta_{\max} \int_\Omega \frac{1}{1 - \mathcal{G}_\alpha(x)} dx < 1 \Rightarrow \rho \int_\Omega \frac{1}{\beta_{\max} - \beta(x)} dx < 1.$$

Thus the criterion for the existence of principal eigenvalue that we gave in (13) is reasonable and comparable with one in nonlocal problems, see Coville 2010 JDE and Shen and Vo 2019 JDE.

Asynchronous Exponential Growth (An Unexpected Discovery)

Definition 7

A C_0−semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space Z has *asynchronous exponential growth* with *intrinsic growth constant* $\lambda_1 \in \mathbb{R}$, if there exists a non-zero finite rank operator P on Z such that $\lim_{t\to\infty} e^{-\lambda_1 t} S(t) = P$, where the limit is in the operator norm topology.

[2] $(s(\mathcal{A}) =)\omega(S) > \omega(T)(= s(\mathcal{B}_1 + \mathcal{C})) \geq \omega_1(T) = \omega_1(S)$.

Asynchronous Exponential Growth (An Unexpected Discovery)

Definition 7

A C_0-semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space Z has *asynchronous exponential growth* with *intrinsic growth constant* $\lambda_1 \in \mathbb{R}$, if there exists a non-zero finite rank operator P on Z such that $\lim_{t\to\infty} e^{-\lambda_1 t}S(t) = P$, where the limit is in the operator norm topology.

Theorem 8 (Thieme, 1998 DCDS, Theorem 2.5)

Let X be a Banach lattice and A resolvent positive, S_0 be the positive C_0-semigroup on $\overline{D(A)}$ generated by A_0, the part of A in X_0. Then S_0 exhibits asynchronous exponential growth iff S_0 is essentially compact $(\omega_1(S_0) < \omega(S_0))$ and $S(A)$ is a first order pole of the resolvent of A.

---

[2] $(s(\mathcal{A}) =)\omega(S) > \omega(T)(= s(\mathcal{B}_1 + \mathcal{C})) \geq \omega_1(T) = \omega_1(S)$.

Asynchronous Exponential Growth (An Unexpected Discovery)

Definition 7

A C_0-semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space Z has *asynchronous exponential growth* with *intrinsic growth constant* $\lambda_1 \in \mathbb{R}$, if there exists a non-zero finite rank operator P on Z such that $\lim_{t\to\infty} e^{-\lambda_1 t}S(t) = P$, where the limit is in the operator norm topology.

Theorem 8 (Thieme, 1998 DCDS, Theorem 2.5)

Let X be a Banach lattice and A resolvent positive, S_0 be the positive C_0-semigroup on $\overline{D(A)}$ generated by A_0, the part of A in X_0. Then S_0 exhibits asynchronous exponential growth iff S_0 is essentially compact $(\omega_1(S_0) < \omega(S_0))$ and $S(A)$ is a first order pole of the resolvent of A.

$$
\begin{aligned}
\frac{1}{1-\mathcal{G}_\alpha} \notin L^1_{loc}(\overline{\Omega}) \quad &\Rightarrow \quad s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})^2 \\
&\Rightarrow \quad S_0 \text{ is essentially compact} \\
&\Rightarrow \quad r(\mathcal{A}) > r_e(\mathcal{A}) \qquad (15)
\end{aligned}
$$

[2] $(s(\mathcal{A}) =)\omega(S) > \omega(T)(= s(\mathcal{B}_1 + \mathcal{C})) \geq \omega_1(T) = \omega_1(S)$.

Generalized Principal Eigenvalue

Due to the lack of usual variational formula, following the idea from Berestyki et. al, 1994 CPAM and 2016 JFA, we define

Definition 9

$$\begin{cases} \lambda_p(\mathcal{A}) := \sup\{\lambda \in \mathbb{R} : \exists\, (0,\phi) \in D(\mathcal{A}) \cap \mathcal{X}_0^{++}, \\ \qquad\qquad s.t. (-\mathcal{A}+\lambda)(0,\phi) \leq (0,0) \text{ in } [0,a^+] \times \overline{\Omega}\}, \\ \lambda_p'(\mathcal{A}) := \inf\{\lambda \in \mathbb{R} : \exists\, (0,\phi) \in D(\mathcal{A}) \cap \mathcal{X}_0^{++}, \\ \qquad\qquad s.t. (-\mathcal{A}+\lambda)(0,\phi) \geq (0,0) \text{ in } [0,a^+] \times \overline{\Omega}\}. \end{cases} \tag{16}$$

Generalized Principal Eigenvalue

Due to the lack of usual variational formula, following the idea from Berestyki et. al, 1994 CPAM and 2016 JFA, we define

Definition 9

$$\begin{cases} \lambda_p(\mathcal{A}) := \sup\{\lambda \in \mathbb{R} : \exists\,(0,\phi) \in D(\mathcal{A}) \cap \mathcal{X}_0^{++}, \\ \qquad\qquad s.t. (-\mathcal{A}+\lambda)(0,\phi) \leq (0,0) \text{ in } [0,a^+] \times \overline{\Omega}\}, \\ \lambda_p'(\mathcal{A}) := \inf\{\lambda \in \mathbb{R} : \exists\,(0,\phi) \in D(\mathcal{A}) \cap \mathcal{X}_0^{++}, \\ \qquad\qquad s.t. (-\mathcal{A}+\lambda)(0,\phi) \geq (0,0) \text{ in } [0,a^+] \times \overline{\Omega}\}. \end{cases} \tag{16}$$

Proposition 4

$\lambda_1(\mathcal{A}) = \lambda_p(\mathcal{A}) = \lambda_p'(\mathcal{A})$ if $\lambda_1(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$.

Limiting Properties without kernel scaling

Theorem 10 (K.-Ruan, 2020)

If $\lambda_1^D(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, then the function $D \to \lambda_1^D(\mathcal{A})$ is continuous on $(0,\infty)$ and satisfies

$$\lambda_1^D(\mathcal{A}) \to \begin{cases} s(\mathcal{B}_1^0 + \mathcal{C}), & as \quad D \to 0^+, \\ -\infty, & as \quad D \to \infty. \end{cases} \tag{17}$$

where

$$\mathcal{B}_1^0(0,f) := \left(-f(0,\cdot) + \int_0^{a^+} \beta(a,\cdot) f(a,\cdot) da, \quad -f' - \mu f \right),$$

for $(0,f) \in D(\mathcal{A})$.

Monotonicity with respect to D.

Remark 11

From Proposition 1, we know that $s(\mathcal{B}_1^0 + \mathcal{C})$ equals to the value α_1 which satisfies

$$\max_{x \in \overline{\Omega}} \int_0^{a^+} \beta(a, x) e^{-\alpha_1 a} \Pi(0, a, x) da = 1.$$

Monotonicity with respect to D.

Remark 11

From Proposition 1, we know that $s(\mathcal{B}_1^0 + \mathcal{C})$ equals to the value α_1 which satisfies

$$\max_{x \in \overline{\Omega}} \int_0^{a^+} \beta(a,x) e^{-\alpha_1 a} \Pi(0,a,x) da = 1.$$

Theorem 12

If $\mu(a,x) = \mu_1(a) + \mu_2(x)$ and $\beta(a,x) \equiv \beta(a)$ and suppose that J is symmetric with respect to each component and in addition, the operator

$$v \to D \left[\int_\Omega J(\cdot - y) v(y) - v \right] - \mu_2 v : X \to X$$

admits a principal eigenvalue, then $D \to \lambda_1^D(\mathcal{A})$ is strictly decreasing.

Continuous Dependence

Proposition 5

Let $m \geq 0, \sigma > 0$,

(i) $\lambda_1(\mathcal{B}^{\mu}_{\sigma,m,\Omega} + \mathcal{C})$ *is Lipschitz continuous with respect to* μ *in* $C([0,a^+],X)$ *if* $\lambda_1(\mathcal{A}_{\sigma,m,\Omega})$ *is the principal eigenvalue of* $\mathcal{A}_{\sigma,m,\Omega}$*. More precisely,*

$$|\lambda_1(\mathcal{B}^{\mu_1}_{\sigma,m,\Omega} + \mathcal{C}) - \lambda_1(\mathcal{B}^{\mu_2}_{\sigma,m,\Omega} + \mathcal{C})| \leq \|\mu_1 - \mu_2\|_{C([0,a^+],X)},$$

for any $\mu_1, \mu_2 \in C([0,a^+],X)$.

(ii) *If* $\Omega_1 \subset \Omega_2$*, then* $\lambda_p(\mathcal{A}_{\sigma,m,\Omega_1}) \leq \lambda_p(\mathcal{A}_{\sigma,m,\Omega_2})$*. If in addition,* $\lambda_1(\mathcal{A}_{\sigma,m,\Omega_1})$ *and* $\lambda_1(\mathcal{A}_{\sigma,m,\Omega_2})$ *are principal eigenvalues of* $\mathcal{A}_{\sigma,m,\Omega_1}$ *and* $\mathcal{A}_{\sigma,m,\Omega_2}$ *respectively, then*

$$|\lambda_p(\mathcal{A}_{\sigma,m,\Omega_1}) - \lambda_p(\mathcal{A}_{\sigma,m,\Omega_2})| \leq C_0|\Omega_2 \setminus \Omega_1|,$$

where $C_0 > 0$ *depends on* a, σ, m, J_σ *and* Ω_2.

Limiting Properties with kernel scaling

Theorem 13 (K.-Ruan, 2020)

If $\lambda_1(\mathcal{A}_{\sigma,m,\Omega}) = s(\mathcal{A}_{\sigma,m,\Omega})$ is the principal eigenvalue of $\mathcal{A}_{\sigma,m,\Omega}$, then

(i) *As $\sigma \to \infty$, there holds*

$$\lambda_1(\mathcal{A}_{\sigma,m,\Omega}) \to \begin{cases} s(\mathcal{B}_1^0 + \mathcal{C}) - D, & m = 0, \\ s(\mathcal{B}_1^0 + \mathcal{C}), & m > 0. \end{cases} \tag{18}$$

(ii) *Suppose, in addition, J is symmetric with respect to each component and $\mu, \beta \in C^{1,4}([0, a^+] \times \overline{\Omega})$. As $\sigma \to 0^+$, there holds*

$$\lambda_1(\mathcal{A}_{\sigma,m,\Omega}) \to s(\mathcal{B}_1^0 + \mathcal{C}), \quad \forall m \in [0, 2).$$

where for $(0, f) \in D(\mathcal{A})$,

$$\mathcal{B}_1^0(0, f) := \left(-f(0, \cdot) + \int_0^{a^+} \beta(a, \cdot) f(a, \cdot) da, \quad -f' - \mu f \right).$$

A Remark

Note that we did not discuss the case when $m = 2$ and $\sigma \to 0$. We conjecture that the principal eigenvalue for age-structured models with nonlocal diffusion converges to the one for age-structured models with Laplace diffusion. Actually, without age-structure, the autonomous nonlocal diffusion operator has a L^2 variational structure which can be used to show the convergence, see Berestycki et al. 2016 JFA and Su et al. 2019 JDE. While for the time-periodic nonlocal diffusion operator, Shen and Xie 2015 DCDS-A used the idea of solution mapping to show the convergence, where they employed the spectral mapping theorem which is not valid in our case either since we have a first order differential operator ∂_a that is unbounded. However, when we add a nonlocal boundary condition to the birth rate β, it can be proved that the semigroup generated by solutions is eventually compact where spectral mapping theorem holds. Thus we can use it to show the desired convergence.

Strong Maximum Principle

Definition 14

We say that $\mathcal{A}$ admits the *strong maximum principle* if for any function $(0, u) \in D(\mathcal{A})$ satisfying

$$\begin{cases} \mathcal{A}(0, u) \leq 0, & \text{in} \quad [0, a^+] \times \Omega, \\ (0, u) \geq 0, & \text{in} \quad [0, a^+] \times \partial\Omega, \end{cases} \tag{19}$$

there must hold $u > 0$ in $[0, a^+] \times \Omega$ unless $u \equiv 0$ in $[0, a^+] \times \Omega$.

Strong Maximum Principle

Definition 14

We say that $\mathcal{A}$ admits the *strong maximum principle* if for any function $(0, u) \in D(\mathcal{A})$ satisfying

$$\begin{cases} \mathcal{A}(0, u) \leq 0, & \text{in} \quad [0, a^+] \times \Omega, \\ (0, u) \geq 0, & \text{in} \quad [0, a^+] \times \partial\Omega, \end{cases} \tag{19}$$

there must hold $u > 0$ in $[0, a^+] \times \Omega$ unless $u \equiv 0$ in $[0, a^+] \times \Omega$.

Theorem 15

If $\lambda_1(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, then $\mathcal{A}$ admits the strong maximum principle if and only if $\lambda_1(\mathcal{A}) < 0$.

Global Dynamics

$$
\begin{cases}
[\frac{\partial}{\partial t} + \frac{\partial}{\partial a}]u(t,a,x) = D\left[\int_\Omega J(x-y)u(t,a,y)dy - u(t,a,x)\right] \\
\qquad\qquad\qquad -\mu(a,x)u(t,a,x) + f(a,x,u(t,a,x)), \\
\qquad\qquad\qquad\qquad\qquad (t,a,x) \in (0,\infty)\times(0,a^+]\times\overline{\Omega}, \\
u(t,0,x) = \int_0^{a^+} \beta(a,x)u(t,a,x)da, \qquad (t,x)\in(0,\infty)\times\overline{\Omega}, \\
u(0,a,x) = u_0(a,x), \qquad (a,x)\in[0,a^+]\times\overline{\Omega}.
\end{cases}
\tag{20}
$$

where f is a KPP type of nonlinearity. A typical example of such a nonlinearity is given as $f(a,x,s) = s(k(a,x)-s)$. In the following we will only consider this case for the convenience.

Steady States

Let's first write down the equation which the equilibrium satisfies,

$$\begin{cases} \frac{\partial u(a,x)}{\partial a} = D\left[\int_\Omega J(x-y)u(a,x)dy - u(a,x)\right] - \mu(a,x)u(a,x) \\ \qquad\qquad +u(a,x)(k(a,x)-u(a,x)), & (a,x)\in(0,a^+]\times\overline{\Omega}, \\ u(0,x) = \int_0^{a^+}\beta(a,x)u(a,x)da, & x\in\overline{\Omega}. \end{cases} \tag{21}$$

where $k(a,x) \leq K$ for any $(a,x) \in [0,a^+] \times \overline{\Omega}$.

Steady States

Let's first write down the equation which the equilibrium satisfies,

$$\begin{cases} \frac{\partial u(a,x)}{\partial a} = D\left[\int_\Omega J(x-y)u(a,x)dy - u(a,x)\right] - \mu(a,x)u(a,x) \\ \qquad\qquad +u(a,x)(k(a,x)-u(a,x)), & (a,x)\in(0,a^+]\times\overline{\Omega}, \\ u(0,x) = \int_0^{a^+}\beta(a,x)u(a,x)da, & x\in\overline{\Omega}. \end{cases} \tag{21}$$

where $k(a,x)\le K$ for any $(a,x)\in[0,a^+]\times\overline{\Omega}$.

Definition 16

We call u is a *supersolution (resp. subsolution)* of (21) if $=$ becomes into $\geq$ (resp. $\leq$) in (21).

Existence, Uniqueness and Stability of (20)

Now let's define the linearized operator $\mathcal{A}_k$ which is obtained by linearizing (21) at $u = 0$,

$$\mathcal{A}_k(0, \phi) := \left(-\phi(0, \cdot) + \int_0^{a^+} \beta(a, \cdot)\phi(a, \cdot)da, \quad -\phi' + L\phi + k\phi \right), \tag{22}$$

for $(0, \phi) \in D(\mathcal{A}_k)$. Denote λ_1^k as the principal eigenvalue of $\mathcal{A}_k$.

Existence, Uniqueness and Stability of (20)

Now let's define the linearized operator $\mathcal{A}_k$ which is obtained by linearizing (21) at $u = 0$,

$$\mathcal{A}_k(0, \phi) := \left(-\phi(0, \cdot) + \int_0^{a^+} \beta(a, \cdot)\phi(a, \cdot)da, \quad -\phi' + L\phi + k\phi \right), \tag{22}$$

for $(0, \phi) \in D(\mathcal{A}_k)$. Denote λ_1^k as the principal eigenvalue of $\mathcal{A}_k$.

Theorem 17

There exists at least a positive nontrivial solution $u^*(a, x)$ of (21) when $\lambda_1^k > 0$. In addition, u^* is unique if β is everywhere positive in $[0, a^+] \times \Omega$. Moreover, the nontrivial equilibrium u^* is stable in the sense of $u(t, a, x) \to u^*(a, x)$ pointwise as $t \to \infty$, where $u(t, a, x)$ is a solution of (20) with initial data $u_0 \geq 0$ and $u_0 \neq 0$.

With Kernel Scaling

Theorem 18

Equation (20) *admits a unique equilibrium* $u^* \in C_{++}([0, a^+], X) \setminus \{0\}$ *that is stable for each* $0 < D \ll 1$ *if* $s(\mathcal{B}_1^0 + \mathcal{C}) > 0$, *where* $s(\mathcal{B}_1^0 + \mathcal{C}) = \alpha_2$ *satisfies* (24).

With Kernel Scaling

Theorem 18

Equation (20) admits a unique equilibrium $u^* \in C_{++}([0, a^+], X) \setminus \{0\}$ that is stable for each $0 < D \ll 1$ if $s(\mathcal{B}_1^0 + \mathcal{C}) > 0$, where $s(\mathcal{B}_1^0 + \mathcal{C}) = \alpha_2$ satisfies (24).

Theorem 19

Equation (20) with kernel scaling defined in (5) admits a unique equilibrium $u^* \in C_{++}([0, a^+], X) \setminus \{0\}$ that is stable in the following cases:

(i) *For each $m > 0$, if $s(\mathcal{B}_1^0 + \mathcal{C}) = \alpha_2 > 0$, then there exists $1 \ll \sigma_1 < \infty$ such that for each $\sigma > \sigma_1$.*

(ii) *Suppose, J is symmetric with respect to each component. For each $m \in [0, 2)$, if $s(\mathcal{B}_1^0 + \mathcal{C}) = \alpha_2 > 0$, then there exists $0 < \sigma_2 \ll 1$ such that for each $\sigma \in (0, \sigma_2)$.*

Asymptotic Behavior of Equilibrium

$$\begin{cases} \frac{\partial v(a,x)}{\partial a} = -\mu(a,x)v(a,x) + v(a,x)(k(a,x) - v(a,x)), & (a,x) \in (0, a^+] \times \overline{\Omega}, \\ v(0,x) = \int_0^{a^+} \beta(a,x)v(a,x)da, & x \in \overline{\Omega}. \end{cases} \quad (23)$$

Lemma 20

Suppose $\alpha_2 > 0$, then for each $x \in \overline{\Omega}$, the equation (23) has a unique positive solution, denoted by $v^*(a,x)$, that is continuous in x, where α_2 satisfies the following equation,

$$\max_{x\in\overline{\Omega}} \int_0^{a^+} \beta(a,x)e^{-\alpha_2 a}\mathcal{K}(0,a,x)da = 1. \quad (24)$$

and $\mathcal{K}(\gamma, a, x) := e^{-\int_\gamma^a (\mu(s,x)-k(s,x))ds}$.

Continued

Theorem 21

If $\alpha_2 > 0$, β is everywhere positive and v^* is from Lemma 20, we have the following asymptotic results,

(i)

$$\lim_{D\to 0^+} u_D^*(a,x) = v^*(a,x), \quad \textit{uniformly in} \quad (a,x)\in[0,a^+]\times\overline{\Omega}, \quad (25)$$

(ii) *If $m\in[0,2)$ and J is symmetric with respect to each component, then*

$$\lim_{\sigma\to 0^+} u_\sigma^*(a,x) = v^*(a,x), \quad \textit{uniformly in} \quad (a,x)\in[0,a^+]\times\overline{\Omega}, \quad (26)$$

(iii) *If $m>0$, then*

$$\lim_{\sigma\to \infty} u_\sigma^*(a,x) = v^*(a,x), \quad \textit{uniformly in} \quad (a,x)\in[0,a^+]\times\overline{\Omega}, \quad (27)$$

where u_σ^* is from Theorem 19.

Summary

(i) We give the sufficient conditions to the existence of the principal eigenvalue and an counterexample where no principal eigenvalue is allowed.

Summary

(i) We give the sufficient conditions to the existence of the principal eigenvalue and an counterexample where no principal eigenvalue is allowed.

(ii) We use generalized principal eigenvalue to characterize the principal eigenvalue and use it to establish the effects of diffusion rate on the principal eigenvalue.

Summary

(i) We give the sufficient conditions to the existence of the principal eigenvalue and an counterexample where no principal eigenvalue is allowed.

(ii) We use generalized principal eigenvalue to characterize the principal eigenvalue and use it to establish the effects of diffusion rate on the principal eigenvalue.

(iii) We establish the strong maximum principle for such age-structured models with nonlocal diffusion.

Summary

(i) We give the sufficient conditions to the existence of the principal eigenvalue and an counterexample where no principal eigenvalue is allowed.

(ii) We use generalized principal eigenvalue to characterize the principal eigenvalue and use it to establish the effects of diffusion rate on the principal eigenvalue.

(iii) We establish the strong maximum principle for such age-structured models with nonlocal diffusion.

(iv) We investigate the existence, uniqueness and stability of such equations with KPP type of nonlinearity.

Summary

(i) We give the sufficient conditions to the existence of the principal eigenvalue and an counterexample where no principal eigenvalue is allowed.

(ii) We use generalized principal eigenvalue to characterize the principal eigenvalue and use it to establish the effects of diffusion rate on the principal eigenvalue.

(iii) We establish the strong maximum principle for such age-structured models with nonlocal diffusion.

(iv) We investigate the existence, uniqueness and stability of such equations with KPP type of nonlinearity.

We expect that such analysis on the principal eigenvalue is helpful to study the traveling wave solutions and spreading speeds of age-structured models with nonlocal diffusion.

Thank you!