Stability of one-dimensional and multi-dimensional fronts in exponentially weighted norms for a class of reaction diffusion equations

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A typical example

Combustion model for a one-dimensional fuel

$$U_t = \partial_{xx} U + Vg(U), \quad U = U(x,t) \text{ temperatura}$$
$$V_t = \epsilon \partial_{xx} V - \kappa V g(U), \quad V = V(x,t) \text{ concentration of unburnt fuel}$$
$$g(U) = \begin{cases} e^{-\frac{1}{U}}, & \text{if } U \geq 0 \\ 0, & \text{if } U < 0 \end{cases} \quad \text{unit reaction rate, } 1 >> \epsilon \geq 0,\ \kappa > 0.$$

$\epsilon = 0$ when the fuel is solid.
κ is the exothermicity, the larger κ is the more fuel one has to burn to achieve a given increase of the temperature.
$U = 0$ background temperature (no reaction).
Traveling combustion front $\phi(\xi) = (U(\xi), V(\xi))$, $\xi = x - ct$, $c > 0$ speed of the front moving to the right. Behind the front $(U, V) = (U_{-\infty}, 0)$ (burnt fuel). Ahead of the front $(U, V) = (0, V_{+\infty})$ (concentration of unburnt fuel $V_{+\infty} > 0$).
We study one- and multidimensional generalizations of this reaction-diffusion system

Overview

Introduction: no formulas, just pictures

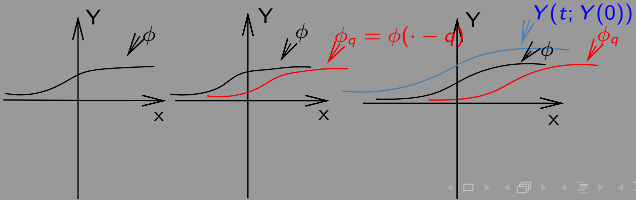
Stable foliations in vicinity of a traveling front for one dimensional reaction diffusion systems

Planar fronts in multidimensional reaction diffusion systems

Brief history

We study *stability* of front solutions of nonlinear equations. For *existence* see [Berestycki, Larrouturou, P.L. Lions], [Berestycki, Nirenberg], [Fiedler, Scheel, Vishik], [Fife], [Hamel, Roquejoffre], [Henry], [Haragus, Scheel], [Kapitula, Promislow], [Morita, Ninomiya], [Rabinowitz], [Sandstede], [Volpert, Volpert, Volpert], [Xin] and many others.

Planar fronts are solutions to partial differential equations that move in a given direction with constant speed without changing their shape and are asymptotic to spatially constant steady-state solutions, the end states. Translations of fronts are also fronts. We prove orbital stability of fronts, that is, show that a small perturbation of a front evolves to a translation of the front



Classical 1-dimensional case

See Bates, Henry, Jones, Pego, Sandstede, Sattinger, Scheel, Volpert, Volpert, Volpert, Weinstein – many many others – classical book by [Volpert^3], newer book by [Kapitula/Promislow])

Let $Y(t, Y(0))$ be the solution to a reaction-diffusion system $Y_t = DY_{xx} + cY_x + R(Y)$ that has a traveling front solution ϕ, that is, $D\phi_{xx} + c\phi_x + R(\phi) = 0$.

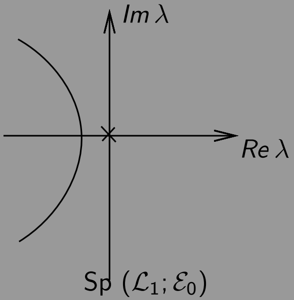
Decompose: Solution = component in the direction of the front + normal to the front,

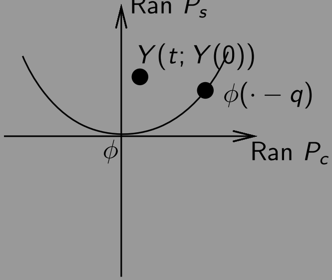
$Y(t, Y(0)) = \phi(\cdot - q(t)) + v(t)$, where $Y(0)$ is close to ϕ.

Linearize at the wave ϕ, let $\mathcal{L}_1$ be the 1-dimensional linear operator obtained by the linearization. Since ϕ_x satisfies $\mathcal{L}_1\phi_x = 0$, the spectrum of $\mathcal{L}_1$ contains 0. Assume 0 is the only unstable spectrum of $\mathcal{L}_1$. Let P_s be the projection on the stable part of the spectrum.

$$\begin{cases} \dot{v} = (\mathcal{L}_1|_{\text{ran } P_s})v + small(v, q) \quad \Rightarrow \quad \|v(t)\|_{\mathcal{E}_0} \leq Ce^{-\nu t} \\ \dot{q} = 0(\text{the eigenvalue}) + small(v, q) \quad \Rightarrow \quad q(t) \to q_* \end{cases}$$

Classical 1-dimensional case in pictures

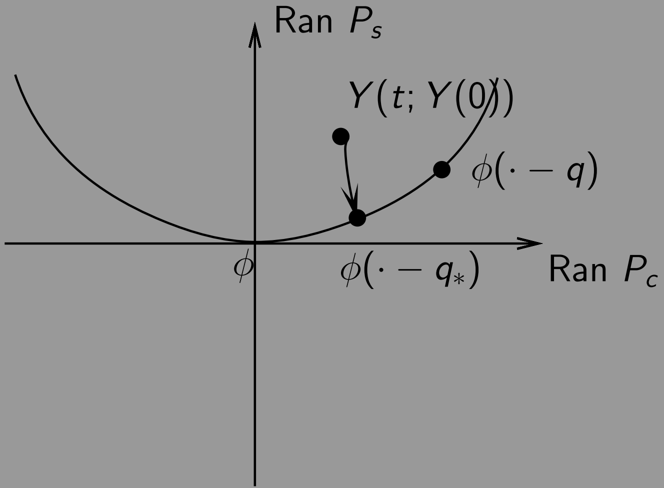




$$\text{Sp}(L_1|\,\text{ran}\,P_c) = \{0\},\ \text{ran}\,P_c = \text{span}\{\phi'\}$$

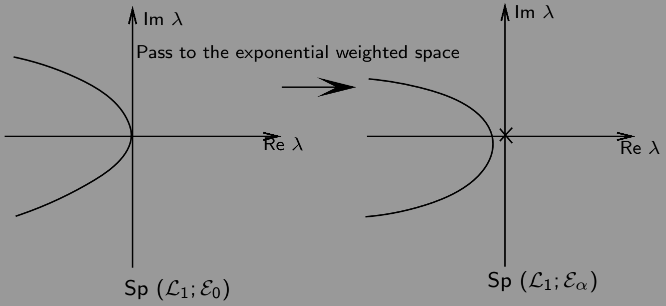
Conclusion for the classical 1-dimensional case

Orbital Stability: $Y(t, Y(0)) \to \phi(\cdot - q_*)$.



More complicated 1-dimensional case

Newer work by many including [Ghazaryan/Latushkin/Schecter]



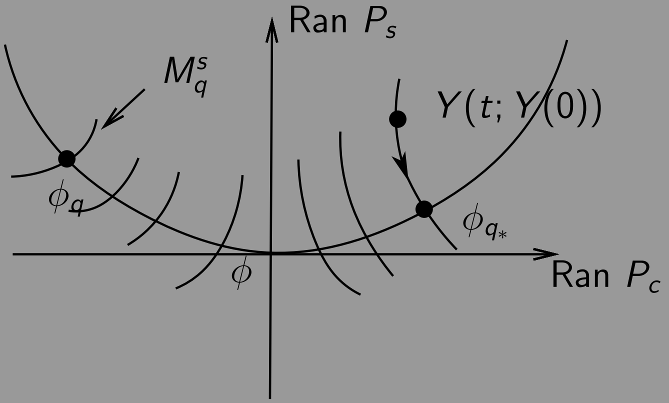
Spectrum is good, nonlinearity is bad, so one needs to pass to the intersection space $\mathcal{E}_0 \cap \mathcal{E}_\alpha$, see [GLS]. Then:

$$\|v(t)\|_{\mathcal{E}_\alpha} \leq Ce^{-\nu t}; \|v(t)\|_{\mathcal{E}_0} \leq C, q(t) \to q_*.$$

Moreover, in appropriated variables $v = (v_1, v_2)$ with $v_1 \in \mathbb{R}^{n_1}$, $v_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, we have $\|v_1(t)\|_{\mathcal{E}_0} \leq C$, $\|v_2(t)\|_{\mathcal{E}_0} \leq Ce^{-\nu t}$.

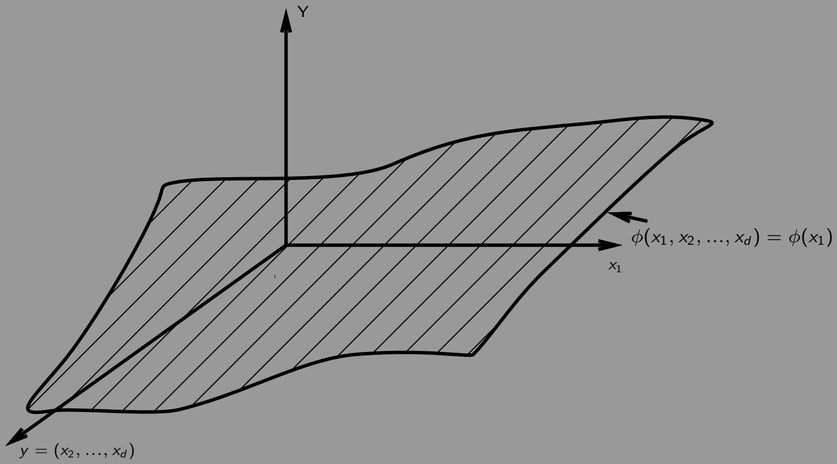
Our current 1-dimensional work

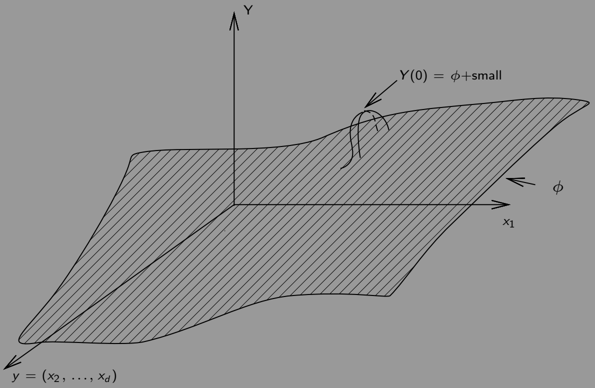
We prove for each q a stable manifold exists through $\phi(\cdot - q)$.



Multidimensional case (earlier work by many in particular by [Kapitula])

$$Y_t = (\partial_{x_1}^2 + \Delta_y)Y + c\partial_{x_1} Y + R(Y)$$





Decompose: solution = component in the direction of the front + transversal to the front

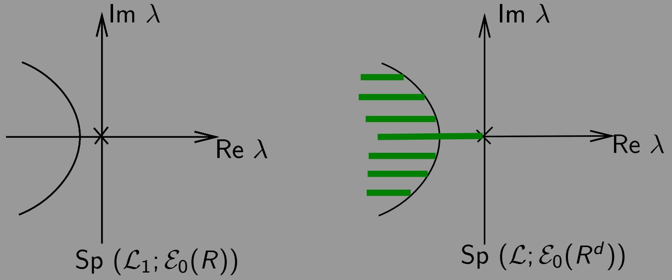
$$Y(t, Y(0))(x_1, y) = \phi(x_1 - q(t, y)) + v(t, x_1, y),$$

$q(t, y)$ is the drift along the front in $y = (x_2, \ldots, x_d)$.

Linearization $\mathcal{L}$ as in [Kapitula] is given by

$$(\mathcal{L}u)(x_1, y) = \big(\mathcal{L}_1 u(\cdot\,, y)\big)(x_1) + \big(\Delta_y u(x_1, \cdot)\big)(y)$$

$$Y_t = (\partial_{x_1}^2 + \Delta_y)Y + c\partial_{x_1} Y + R(Y)$$



Algebraic decay in earlier work [Kapitula]

$$\begin{cases} \dot{v} = \mathcal{L}|_{\mathrm{ran}(P_s(x)\otimes I_y)} v + small(v,q) \\ \dot{q} = \Delta_y q + small(v,q). \end{cases}$$

$$\|e^{t\Delta_y}\|_{L^1(\mathbb{R}^{d-1})\to H^k(\mathbb{R}^{d-1})} \leq \frac{C}{(1+t)^{(d-1)/4}}$$

$$\Rightarrow \|q(t)\|_{H^k(\mathbb{R}^{d-1})} \to 0 \quad \text{algebraically as} \quad t\to\infty$$

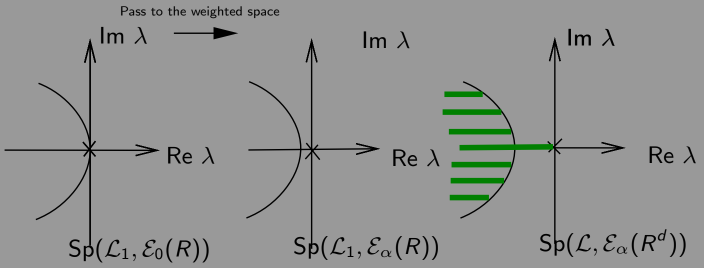
$$\|v(t)\|_{\mathcal{E}_0} \to 0 \quad \text{algebraically as} \quad t\to\infty.$$

Since the drift along the front fades away

$$\Rightarrow Y(t, Y(0)) \to \phi \quad \text{as} \quad t\to\infty \quad \text{algebraically}$$

A more complicated case (our current work)

Linearization $(\mathcal{L}_\alpha u)(x_1, y) = \big(\mathcal{L}_{1,\alpha} u(\cdot\,, y)\big)(x_1) + \big(\Delta_y u(x_1, \,\cdot)\big)(y)$ in the current work



Spectrum is good, nonlinearity is bad, and we pass to the intersection space $\mathcal{E}_0 \cap \mathcal{E}_\alpha$. We prove that

$$\|v(t)\|_{\mathcal{E}_0} \leq C$$
$$\|v(t)\|_{\mathcal{E}_\alpha} \leq \frac{C}{\text{poly. of } t} \to 0$$
$$\|q(t)\|_{H^k(\mathbb{R}^{d-1})} \leq \frac{C}{\text{poly. of } t} \to 0.$$

Moreover, in appropriate variables $v = (v_1, v_2)$,

$$\|v_1(t)\|_{\mathcal{E}_0} \leq C;$$
$$\|v_2(t)\|_{\mathcal{E}_\alpha} \leq \frac{C}{\text{poly. of } t} \to 0,$$

as $t \to \infty$.

CHAPTER 1: A class of 1-dimensional reaction diffusion systems

Consider the system of reaction diffusion equations,

$$Y_t(t,x) = D\partial_{xx}Y(t,x) + R(Y(t,x)),\ Y \in \mathbb{R}^n,\ x \in \mathbb{R},\ t > 0, \quad (1)$$

where $D = diag(d_1, \cdots, d_n)$ with all $d_i \geq 0$, and the function $R(\cdot)$ is smooth and satisfies some additional special properties listed later. A typical example that we have in mind is the following system from solid combustion for $Y = (U, V)$:

$$\begin{cases} U_t(t,x) = \partial_{xx}U(t,x) + V(t,x)g(U(t,x)),\ U, V \in \mathbb{R}, \\ V_t(t,x) = \epsilon\partial_{xx}V(t,x) - \kappa V(t,x)g(U(t,x)),\ x \in \mathbb{R}, \end{cases} \quad (2)$$

where

$$g(U) = \begin{cases} e^{-\frac{1}{U}} & \text{if } U > 0; \\ 0 & \text{if } U \leq 0, \end{cases} \quad (3)$$

Hypotheses

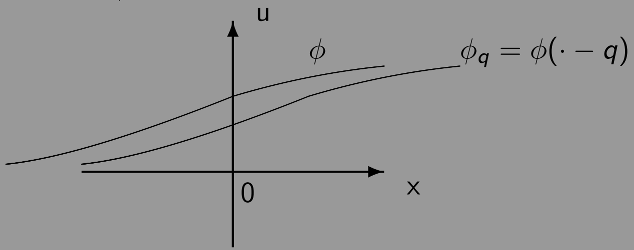
Passing to the moving coordinate frame $\xi = x - ct$ and redenoting ξ again by x, we arrive at the nonlinear equation

$$Y_t = DY_{xx} + cY_x + R(Y), \qquad x \in \mathbb{R},\ t \geq 0. \tag{4}$$

Assume that system (4) admits a traveling wave solution $\phi(x)$ that converges to the end states $\phi_\pm$ as $x \to \pm\infty$ exponentially; i.e.,

$$\begin{aligned} |\phi(x) - \phi_-| &\leq Ce^{-\omega_- x}, \qquad x \leq 0, \\ |\phi(x) - \phi_+| &\leq Ce^{-\omega_+ x}, \qquad x \geq 0, \end{aligned} \tag{5}$$

for some $\omega_- < 0 < \omega_+$ and $C > 0$. Without loss of generality, we also assume that $\phi_- = 0$.



We study the system on the unweighted space $\mathcal{E}_0 = H^1(\mathbb{R})$ since it is closed under multiplication, and denote the norm on $\mathcal{E}_0$ by $\|\cdot\|_0$. Let $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$. We say that $\gamma_\alpha : \mathbb{R} \to \mathbb{R}$ is a weight function of class α if γ_α is C^2, $\gamma_\alpha(x) > 0$ for all $x \in \mathbb{R}$, and $\gamma_\alpha(x) = e^{\alpha_- x}$ for $x \leq -x_0$ and $\gamma_\alpha(x) = e^{\alpha_+ x}$ for $x \geq x_0$ for some $x_0 > 0$. We assume that $0 < \alpha_- < -\omega_-$ and $0 \leq \alpha_+ < \omega_+$, where $\omega_\pm$ are the exponents that control the decay of ϕ to $\phi_\pm$. Given such a pair $\alpha = (\alpha_-, \alpha_+)$, we introduce the weighted space $\mathcal{E}_\alpha = \{u : \mathbb{R} \to \mathbb{R}^n : \gamma_\alpha u \in \mathcal{E}_0\}$ with the norm $|u|_\alpha = |\gamma_\alpha u|_0$. The intersection space $\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ is endowed with the norm

$$|u|_{\mathcal{E}} = \max\{|u|_0, |u|_\alpha\}$$

Example: $\gamma_\alpha(x) = e^{\alpha x}$, $\mathcal{E}_0 = H^1(\mathbb{R})$, $\mathcal{E}_\alpha = \{u : e^{\alpha x} u \in H^1(\mathbb{R})\}$. Isometry $M_\alpha : \mathcal{E}_\alpha \to \mathcal{E}_0 : u \mapsto e^{\alpha x} u$. The operator $\partial_{x,\alpha} : u \mapsto u'$ on $\mathcal{E}_\alpha$ is similar via $M_\alpha \partial_{x,\alpha} M_\alpha^{-1} = \partial_{x,0} - \alpha$ to $\partial_{x,0} - \alpha$, where $\partial_{x,0} : u \mapsto u'$, because $\partial_{x,\alpha} M_\alpha^{-1} u = (e^{-\alpha x} u)' = e^{-\alpha x}(u' - \alpha)$.

We further assume that the nonlinear term R in $Y_t = DY_{xx} + cY_x + R(Y)$ has the following product structure: The nonlinear term R belongs to $C^4(\mathbb{R}^n, \mathbb{R}^n)$. In appropriate variables $Y = (U, V)^T$ with $U \in \mathbb{R}^{n_1}$, $V \in \mathbb{R}^{n_2}$ and $n_1 + n_2 = n$, we have

$$R(U, 0) = (A_1 U, 0) \tag{6}$$

for a constant $n_1 \times n_1$ matrix A_1. In other words, we suppose that

$$R(U, V) = \begin{pmatrix} A_1 U + R_1(U, V) \\ R_2(U, V) \end{pmatrix} = \begin{pmatrix} A_1 U + \tilde{R}_1(U, V)V \\ \tilde{R}_2(U, V)V \end{pmatrix},$$

where the maps R_j belong to $C^3(\mathbb{R}^n, \mathbb{R}^{n_j})$ and satisfy $R_j(U, 0) = 0$ for $j \in \{1, 2\}$ and $U \in \mathbb{R}^{n_1}$. Note that condition (6) yields $R(0, 0) = R(\phi_-) = 0$. We also split

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad D_1 = diag(d_1, \dots, d_{n_1}), \quad D_2 = diag(d_{n_1+1}, \dots, d_n).$$

Let $q \in \mathbb{R}$. We write $\phi_q(x) = \phi(x - q)$ for the shifted wave. Linearizing $Y_t = DY_{xx} + cY_x + R(Y)$ at ϕ_q, we arrive at

$$Y_t = L_q Y + F_q(Y), \text{ where } L_q Y = DY_{xx} + cY_x + \partial_Y R(\phi_q) Y. \quad (7)$$

Here, the nonlinear term $F_q : \mathbb{R}^n \to \mathbb{R}^n$ is written as

$$F_q(Y) = \int_0^1 \left(\partial_Y R(\phi_q + tY) - \partial_Y R(\phi_q) \right) Y dt. \quad (8)$$

The linearization of $Y_t = DY_{xx} + cY_x + R(Y)$ at $\phi_- = (0,0)^T$ is

$$Y_t = L^- Y + G(Y), \text{ where } L^- Y = DY_{xx} + cY_x + \partial_Y R(0) Y \quad (9)$$

and $G : \mathbb{R}^n \to \mathbb{R}^n$; $G(Y) = R(Y) - \partial_Y R(0) Y$.

Linearization $L_q Y = DY_{xx} + cY_x + \partial_Y R(\phi_q) Y$
We will impose conditions on L_0 at $q = 0$; i.e., on the linearization at the original traveling wave ϕ. We further consider L_q for $|q| \le q_0$ with some $q_0 > 0$.
Linearization $L^- Y = DY_{xx} + cY_x + \partial_Y R(0) Y$

$$\partial_Y R(0,0) = \begin{pmatrix} A_1 & \partial_V R_1(0,0) \\ 0 & \partial_V R_2(0,0) \end{pmatrix}, \qquad L^- = \begin{pmatrix} L^{(1)} & \partial_V R_1(0,0) \\ 0 & L^{(2)} \end{pmatrix} \tag{10}$$

with the differential expressions

$$L^{(1)} U = D_1 U_{xx} + cU_x + A_1 U,$$
$$L^{(2)} V = D_2 V_{xx} + cV_x + \partial_V R_2(0,0) V.$$

Assumptions on linearization $L_q Y = DY_{xx} + cY_x + \partial_Y R(\phi_q)Y$:
We assume that there exists $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$ such that

(a) $\sup\{\mathrm{Re}\lambda : \lambda \in \mathsf{Sp}_{\mathsf{ess}}(\mathcal{L}_{0,\alpha})\} < 0$ for the differential operator on $\mathcal{E}_\alpha$ generated by L_0.

(b) The only element of $\mathsf{Sp}(\mathcal{L}_{0,\alpha})$ in $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq 0\}$ is a simple eigenvalue at $\lambda = 0$ with ϕ' being the respective eigenfunction.

We let P_q^c denote the spectral projection for $\mathcal{L}_{q,\alpha}$ in $\mathcal{E}_\alpha$ onto $\ker \mathcal{L}_{q,\alpha} = \mathrm{span}\{\phi_q'\}$ and the complementary projection by $P_q^s = I - P_q^c$. Denote by $\{T_q(t)\}_{t \geq 0}$ the semigroup generates by $\mathcal{L}_q$, this implies $\|T_q(t)P_q^s\|_{\mathcal{B}(\mathcal{E}_\alpha)} \leq Ce^{-\nu t}$.

Assumptions on linearization

$$L^{-}Y = DY_{xx} + cY_x + \partial_Y R(0)Y = \begin{pmatrix} L^{(1)} & \partial_V R_1(0,0) \\ 0 & L^{(2)} \end{pmatrix} Y :$$

Denote by $\{S_1(t)\}_{t\geq 0}$, $\{S_2(t)\}_{t\geq 0}$ the semigroups generated by $L^{(1)}U = D_1 U_{xx} + cU_x + A_1 U$, $L^{(2)}V = D_2 V_{xx} + cV_x + \partial_V R_2(0,0)V$ on $\mathcal{E}_0$ for the decomposition $Y = (V, U)$ and assume the following: The strongly continuous semigroup $\{S_1(t)\}_{t\geq 0}$ is bounded and the semigroup $\{S_2(t)\}_{t\geq 0}$ is uniformly exponentially stable on $\mathcal{E}_0$:

$$\|S_1(t)\|_{\mathcal{B}(\mathcal{E}_0)} \leq C, \qquad \|S_2(t)\|_{\mathcal{B}(\mathcal{E}_0)} \leq Ce^{-\rho t}$$

for some $\rho > 0$ and all $t \geq 0$.
This also implies (a lemma):

$$\|S(t)\|_{\mathcal{B}(\mathcal{E}_0)} \leq C, \text{ for all } \; t \geq 0. \tag{11}$$

$$\sup_{|q|\leq q_0} \sup_{t\geq 0} \|T_q(t)\|_{\mathcal{B}(\mathcal{E})} < \infty, . \tag{12}$$

Nonlinearity

$Y_t = L_q Y + F_q(Y)$, $F_q(Y) = \int_0^1 (\partial_Y R(\phi_q + tY) - \partial_Y R(\phi_q))\, Y dt$.
Assume that $\alpha = (\alpha_-, \alpha_+)$ satisfies $0 < \alpha_- < -\omega_-$ and $0 \leq \alpha_+ < \omega_+$, and that the nonlinearity $R \in C^4(\mathbb{R}^n, \mathbb{R}^n)$ fulfills $R(U, 0) = (A_1 U, 0)$. Let $\delta_1 > 0$ and choose a radius $\delta \in (0, \delta_1]$. Then for all functions $y = (u, v)$ and $\bar{y} = (\bar{u}, \bar{v})$ from $\mathcal{E}$ with $|y|_{\mathcal{E}}, |\bar{y}|_{\mathcal{E}} \leq \delta$ the estimates

$$|F_q(y)|_0 \leq C|y|_0 \,(|y|_\alpha + |v|_0), \tag{13}$$

$$|F_q(y)|_\alpha \leq C|y|_0 \,|y|_\alpha, \tag{14}$$

$$|F_q(y) - F_q(\bar{y})|_0 \leq C\big(|y - \bar{y}|_0 \,(|y|_\alpha \tag{15}$$

$$+ |\bar{y}|_\alpha) + |y - \bar{y}|_0 \,|v|_0 + |\bar{y}|_0 \,|v - \bar{v}|_0\big), \tag{16}$$

$$|F_q(y) - F_q(\bar{y})|_\alpha \leq |y - \bar{y}|_\alpha \,(|y|_0 + |\bar{y}|_0) \tag{17}$$

are true, where $C = C(\delta_1, q_0)$ and $|q| \leq q_0$.

The Lyapunov-Perron operator

We next establish basic properties of the Lyapunov-Perron operator $\Phi_q(y, z_0)$ for $Y_t = L_q Y + F_q(Y)$ defined by

$$\begin{aligned}\Phi_q(y, z_0)(t) = T_q(t) P_q^{\mathrm{s}} z_0 + \int_0^t T_q(t-\tau) P_q^{\mathrm{s}} F_q(y(\tau)) d\tau \\ - \int_t^\infty P_q^{\mathrm{c}} F_q(y(\tau)) d\tau,\end{aligned} \tag{18}$$

where $|q| \leq q_0$ and $z_0 \in \mathcal{E}_0 \cap \mathcal{E}_\alpha = \mathcal{E}$ satisfies

$$|z_0|_{\mathcal{E}} = \max\{|z_0|_0, |z_0|_\alpha\} \leq \delta_0, \quad \text{for some} \quad \delta_0 > 0. \tag{19}$$

For continuous $y = (u, v) : \mathbb{R} \to \mathcal{E}_{\mathcal{E}} = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ we define the norms

$$\|y\|_{\omega,\alpha} = \sup_{t \geq 0} e^{\omega t} |y(t)|_\alpha, \ \|y\|_{0,0} = \sup_{t \geq 0} |y(t)|_0, \ \|v\|_{\omega,0} = \sup_{t \geq 0} e^{\omega t} |v(t)|_0,$$

Here we have to modify these exponents such that $0 < \omega < \rho < \nu$. Let $\delta > 0$. Then $\mathbb{B}_\delta(\| \cdot \|)$ is the set of continuous functions $y : \mathbb{R} \to \mathcal{E}_0 \cap \mathcal{E}_\alpha$ such that

$$\|y\| := \max\left(\|y\|_{\omega,\alpha}, \|y\|_{0,0}, \|v\|_{\omega,0}\right) \leq \delta. \tag{20}$$

Properties of Lyapunov-Perron operator

$$\Phi_q(y,z_0)(t) = T_q(t)P_q^{\mathrm{s}}z_0 + \int_0^t T_q(t-\tau)P_q^{\mathrm{s}}F_q(y(\tau))d\tau$$
$$- \int_t^\infty P_q^{\mathrm{c}}F_q(y(\tau))d\tau, \quad Y_t = L_qY + F_q(Y). \tag{21}$$

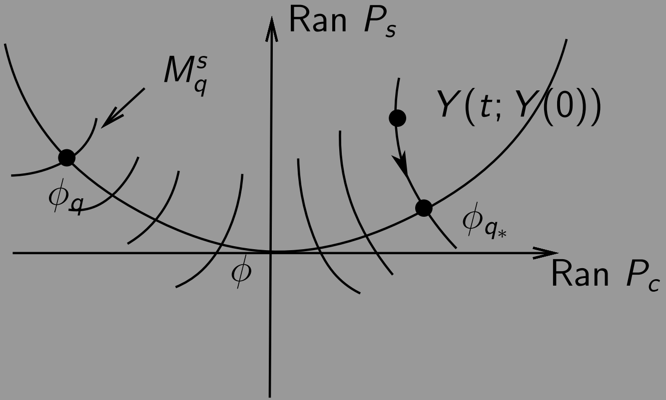
($\mathcal{L}_q$ generates $\{T_q(t)\}$, $\ker(\mathcal{L}_q) = \operatorname{ran} P_q^{\mathrm{c}}$ thus $T_q(t-\tau)P_q^{\mathrm{c}} = P_q^{\mathrm{c}}$) Take $q_0 > 0$. Let $\delta > 0$ and $\delta_0 = \delta_0(\delta) > 0$ be small enough. For each $z_0 \in \mathbb{B}_{\delta_0}(|\cdot|_{\mathcal{E}})$ the Lyapunov-Perron operator $y \mapsto \Phi_q(y,z_0)$ leaves $\mathbb{B}_\delta(\|\cdot\|)$ invariant and is a strict contraction on this ball for all $|q| \le q_0$. Moreover, for the norm $\|\cdot\|$ defined in (20) one has

$$\|\Phi_q(y,z_0) - \Phi_q(\bar{y},\bar{z}_0)\| \le C|z_0 - \bar{z}_0|_{\mathcal{E}} + C\delta\|y-\bar{y}\| \tag{22}$$

for some $C > 0$ and all $z_0, \bar{z}_0 \in \mathbb{B}_{\delta_0}(|\cdot|_{\mathcal{E}})$, $y, \bar{y} \in \mathbb{B}_\delta(\|\cdot\|)$, and $|q| \le q_0$.

Stable manifold

We will now foliate a small neighborhood of ϕ by stable manifolds $\mathcal{M}_q^s$ going through ϕ_q.



Stable manifold

For a small $q_0 > 0$ and each $q \in [-q_0, q_0]$, we now construct a function $\mathbf{m}_q : \operatorname{ran}(P_q^s) \to P_q^c$ whose graph contains ϕ_q and it is a stable manifold $\mathcal{M}_q^s$ for the system $Y_t = DY_{xx} + cY_x + R(Y)$. We further prove that the sets $\mathcal{M}_q^s$ satisfy the standard properties of stable manifolds and that they foliate a small neighborbood of ϕ. Let $\delta, \delta_0 > 0$ be sufficiently small and $q_0 > 0$. Take $|q| \leq q_0$ and $z_0 \in \operatorname{ran}(P_q^s) \cap \mathbb{B}_{\delta_0}(|\cdot|_{\mathcal{E}})$. There exists a unique function $y_{z_0}^q : \mathbb{R}_+ \to \mathcal{E}$ which belongs to $\mathbb{B}_\delta(\|\cdot\|)$ and is a fixed point of the Lyapunov-Perron operator $\Phi_q(\cdot, z_0)$; that is, for $t \geq 0$,

$$
\begin{aligned}
y_{z_0}^q(t) &= T_q(t)z_0 + \int_0^t T_q(t-\tau)P_q^s F_q(y_{z_0}^q(\tau))d\tau - \int_t^\infty P_q^c F_q(y_{z_0}^q(\tau))d\tau \\
&= T_q(t)\big[z_0 - \int_0^\infty P_q^c F_q(y_{z_0}^q(\tau))d\tau\big] + \int_0^t T_q(t-\tau)F_q(y_{z_0}^q(\tau))d\tau.
\end{aligned}
$$

We define the function $\mathbf{m}_q : \operatorname{ran}(P_q^s) \cap \mathbb{B}_{\delta_0}(|\cdot|_{\mathcal{E}}) \to \operatorname{ran}(P_q^c)$ by

$$\mathbf{m}_q(z_0) = -\int_0^\infty P_q^c F_q(y_{z_0}^q(\tau))d\tau. \tag{23}$$

The fixed point $y = y_{z_0}^q$ of the Lyapunov-Perron operator contained in $\mathbb{B}_\delta(\|\cdot\|)$ satisfies $e^{\omega t}|y(t)|_\alpha \leq \delta$, $|y(t)|_0 \leq \delta$ and the equation

$$y(t) = T_q(t)y(0) + \int_0^t T_q(t-\tau)F_q(y(\tau))d\tau, \qquad t \geq 0. \tag{24}$$

For a number $\eta > 0$ to be fixed below, the stable manifold $\mathcal{M}_q^s$ is then defined as the graph of $\mathbf{m}_q(\cdot)$ shifted to ϕ_q by

$$\mathcal{M}_q^s = \{\phi_q + z_0 + \mathbf{m}_q(z_0) : z_0 \in \operatorname{ran}(P_q^s) \cap \mathbb{B}_{\delta_0}(|\cdot|_{\mathcal{E}})\} \cap (\phi + \mathbb{B}_\eta(|\cdot|_{\mathcal{E}})),$$

where $|q| \leq q_0$ and $\phi + \mathbb{B}_\eta(|\cdot|_{\mathcal{E}})$ is the closed ball in $\mathcal{E} = \mathcal{E}_\alpha \cap \mathcal{E}_0$ with radius η and centered at the original traveling wave ϕ.

Theorem

Let $q_0, \delta, \delta_0, \eta > 0$ *be sufficiently small,* $|q| \le q_0$*, and* $0 < \omega < \rho < \nu$*. Then the ball* $\phi + \mathbb{B}_\eta(|\cdot|_\mathcal{E})$ *is foliated by the stable manifolds* $\mathcal{M}_q^s$ *for the nonlinear equation* $Y_t = DY_{xx} + cY_x + R(Y)$ *and the following assertions hold.*

(i) *Each* $\mathcal{M}_q^s$ *is a Lipschitz manifold in* $\mathcal{E}$*. If* $Y(0) \in \mathcal{M}_q^s$ *and the mild solution* $Y(t; Y(0))$ *of* $Y_t = DY_{xx} + cY_x + R(Y)$ *belongs to* $\phi + \mathbb{B}_\eta(|\cdot|_\mathcal{E})$ *for some* $t \ge 0$*, then* $Y(t; Y(0))$ *is contained in* $\mathcal{M}_q^s$*.*

(ii) *For each* $Y(0) \in \mathcal{M}_q^s$ *there exists a solution* $Y(t; Y(0))$ *of* $Y_t = DY_{xx} + cY_x + R(Y)$ *which exists for all* $t \ge 0$ *and satisfies* $|Y(t; Y(0)) - \phi_q|_\mathcal{E} \le \delta$ *as well as*

(a) $|Y(t; Y(0)) - \phi_q|_\alpha \le Ce^{-\omega t}|Y(0) - \phi_q|_\mathcal{E}$,

(b) $|\pi_1(Y(t; Y(0)) - \phi_q) - U_q|_0 \le C|Y(0) - \phi_q|_\mathcal{E}$,

(c) $|\pi_2(Y(t; Y(0)) - \phi_q) - V_q|_0 \le Ce^{-\omega t}|Y(0) - \phi_q|_\mathcal{E}$

for all $t \ge 0$*. Here,* $\phi_q = (U_q, V_q) = \phi(\cdot - q)$ *is the shifted traveling wave,* $\pi_1 : Y = (U, V) \to U$*, and* $\pi_2 : Y = (U, V) \to V$*.*

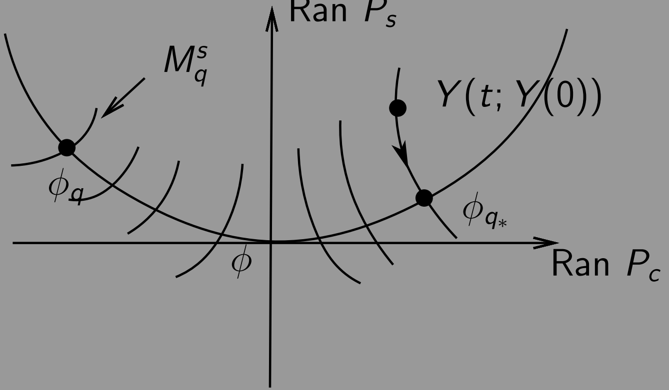
(Continued)

Theorem

(iii) *If* $Y(t;Y(0))$, $t \geq 0$, *is a mild solution of* $Y_t = DY_{xx} + cY_x + R(Y)$ *with* $Y(0) \in \phi + \mathbb{B}_\eta(|\cdot|_\mathcal{E})$ *that satisfies properties (a)–(c) in item (ii), then* $Y(0)$ *belongs to* $\mathcal{M}^s_q$.

(iv) *For* $q \neq \bar{q}$, *we have* $\mathcal{M}^s_q \cap \mathcal{M}^s_{\bar{q}} = \emptyset$. *Moreover,* $\phi + \mathbb{B}_\eta(|\cdot|_\mathcal{E}) = \bigcup_{|q|\leq q_0} \mathcal{M}^s_q$.

(v) *The map* $[-q_0, q_0] \to \mathrm{ran}(P^c_q)$; $q \mapsto \mathbf{m}_q(P^s_q z_0)$, *is Lipschitz for each* $z_0 \in \mathbb{B}_{\delta_0}(|\cdot|_\mathcal{E})$.



For each $Y(0) \in \phi + \mathbb{B}_\eta(|\cdot|_{\mathcal{E}})$ there exists exactly one shift $q \in [-q_0, q_0]$ such that $Y(0) \in \mathcal{M}^s_q$.

The stability result: the idea of the proof, no formulas

We show that $\|Y(t)\|_{\mathcal{E}_\alpha}$ exp decay, $\|Y(t)\|_{\mathcal{E}_0}$ bounded for $Y_t = DY_{xx} + cY_x + R(Y)$, $Y = (U, V)$. As $P^{\rm c}$ is one dimensional thus easy, consider $P^{\rm s}Y$. Use bootstrap: $0 < \gamma < \delta$ if $\|Y(0)\|_{\mathcal{E}} \leq \gamma$ then $\|Y(t)\|_{\mathcal{E}} \leq \delta$ for all $t < T_{\max}(\gamma, \delta) \leq \infty$.
1) As long as $\|Y(t)\|_{\mathcal{E}_0}$ is small, $\|Y(t)\|_{\mathcal{E}_\alpha}$ exp decay by Gronwall's because $T(t)P^{\rm s}$ exp decay in $\mathcal{B}(\mathcal{E}_\alpha)$,

$$P^{\rm s}Y(t) = T(t)P^{\rm s}Y(0) + \int_0^t T(t-s)P^{\rm s}O(\|Y(s)\|_{\mathcal{E}_0} \times \|Y(s)\|_{\mathcal{E}_\alpha})ds$$

2) As long as $\|Y(t)\|_{\mathcal{E}_\alpha}$ exp decay, $\|Y(t)\|_{\mathcal{E}_0}$ is small by Gronwall's because $S_1(t)P^{\rm s}$ is bounded and $S_2(t)P^{\rm s}$ exp decay in $\mathcal{B}(\mathcal{E}0)$,

$$U(t) = S_1(t)U(0) + \int_0^t S_1(t-s)O((\|U(s)\|_{\mathcal{E}_0} + \|V(s)\|_{\mathcal{E}_0}) \times \|Y(s)\|_{\mathcal{E}_\alpha})ds$$

$V(t) = S_2(t)V(0) + \int_0^t S_2(t-s)O(\|V(s)\|_{\mathcal{E}_0} \times \|Y(s)\|_{\mathcal{E}_\alpha})ds$
It follows that $T_{\max}(\gamma, \delta) = \infty$

Chapter Two: Multidimensional model case

Consider the combustion system of two equations in $\mathbb{R}$,

$$\begin{cases} U_t(t,x) = \Delta_x U(t,x) + V(t,x)g(U(t,x)),\ U, V \in \mathbb{R}^n,\ \ n \geq 2 \\ V_t(t,x) = \Delta_x V(t,x) - \kappa V(t,x)g(U(t,x)),\ x \in \mathbb{R}^d, \end{cases} \tag{25}$$

where

$$g(U) = \begin{cases} e^{-\frac{1}{U}} & \text{if } U > 0; \\ 0 & \text{if } U \leq 0, \end{cases} \tag{26}$$

Multidimensional stability of a planar front

We consider a general reaction-diffusion system

$$u_t(t,x) = \Delta_x u(t,x) + f(u(t,x)), \tag{27}$$

where $u \in \mathbb{R}^n$, $x \in \mathbb{R}^d$, $t \geq 0$, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is smooth.
Given the vector $e = (1, 0, ..., 0) \in \mathbb{S}^d$, we will make a change of variable $z = e \cdot x - ct$ for some velocity $c > 0$. Redenoting again $x = (z, x_2, ..., x_d)$, we arrive at the equation

$$u_t = \Delta_x u + cu_z + f(u), \tag{28}$$

where $\Delta_x = \partial_z^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2 = \partial_z^2 + \Delta_y$.
A traveling wave solution $\phi = \phi(z)$ for system $u_t(t,x) = \Delta_x u(t,x) + f(u(t,x))$ is a smooth function of $z \in \mathbb{R}$ that is a time independent solution of (28) and satisfies

$$0 = \partial_{zz}\phi + c\partial_z\phi + f(\phi). \tag{29}$$

Linearizing $u_t = \Delta_x u + cu_z + f(u)$ about ϕ, we obtain the variable coefficients expression $L = \Delta_x + c\partial_z + \partial_u f(\phi)$.
We need the spectral information about $\mathcal{L}$ associated with L on

$$\mathcal{E}_0 = H^k(\mathbb{R}^d)^n = H^k\big(\mathbb{R}; H^k(\mathbb{R}^{d-1}; \mathbb{C}^n)\big) = H^k\big(\mathbb{R}^{d-1}; H^k(\mathbb{R}; \mathbb{C}^n)\big),$$

we will assume $k \geq [\frac{d+1}{2}]$ throughout. Thus we decompose $\mathcal{L}$ as

$$(\mathcal{L}u)(z, y) = \big(\mathcal{L}_1 u(\cdot\,, y)\big)(z) + \big(\Delta_y u(z, \cdot)\big)(y)$$

where $\mathcal{L}_1$ is associated with the one-dimensional differential variable coefficients expression $L_1 = \partial_z^2 + c\partial_z + \partial_u f(\phi)$ that depends only on z, and $\Delta_y = (\partial_{x_2}^2 + \cdots + \partial_{x_d}^2)$.

We assume that there exist constant solutions $\phi_\pm \in \mathbb{R}^n$ of $u_t(t,x) = \Delta_x u(t,x) + f(u(t,x))$ so that $f(\phi_\pm) = 0$ and there exist constants $K > 0$ and $\omega_- < 0 < \omega_+$ such that
$||\phi(z) - \phi_-||_{\mathbb{R}^n} \leq K e^{-\omega_- z}$ for $z \leq 0$,
$||\phi(z) - \phi_+||_{\mathbb{R}^n} \leq K e^{-\omega_+ z}$ for $z \geq 0$.
On $\mathcal{E}_0^n$ the essential spectrum of $L_1 = \partial_z^2 + c\partial_z + \partial_u f(\phi)$ may touch the imaginary axis. To fix this, we introduce a class of weight functions of exponential type. Let $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$. We call $\gamma_\alpha : \mathbb{R} \to \mathbb{R}$ a weight function of class α if $0 < \gamma_\alpha(z)$ for all $z \in \mathbb{R}$, the function $\gamma_\alpha \in C^{k+3}(\mathbb{R})$, and

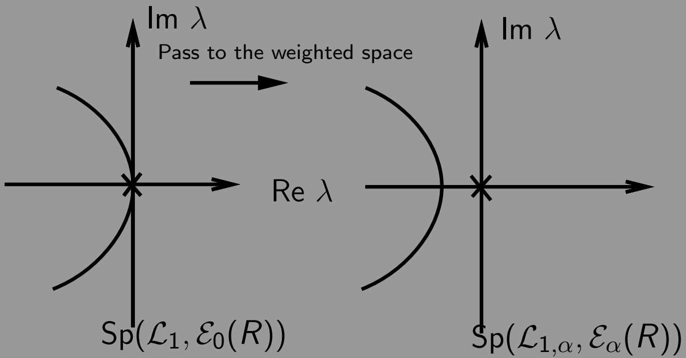
$$\gamma_\alpha(z) = \begin{cases} e^{\alpha_- z}, & \text{for large negative } z, \\ e^{\alpha_+ z}, & \text{for large positive } z. \end{cases} \qquad (30)$$

Following the setting in [GLS], we will always assume that

$$0 < \alpha_- < -\omega_- \quad \text{and} \quad 0 \leq \alpha_+ < \omega_+.$$

For a fixed weight function γ_α, let $\mathcal{E}_\alpha = \{u : \gamma_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})} u \in \mathcal{E}_0\}$, with the norm $\|u\|_\alpha = \|\gamma_\alpha u\|_0$. Note that by the definition of $\mathcal{E}_\alpha$, we can represent the weighted space $\mathcal{E}_\alpha$ by $H^k_\alpha\big(\mathbb{R}; H^k(\mathbb{R}^{d-1}; \mathbb{C}^n)\big)$. Here, for a function $u = u(z, y)$ we denote by $(\gamma_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})})u$ the function of (z, y) defined by

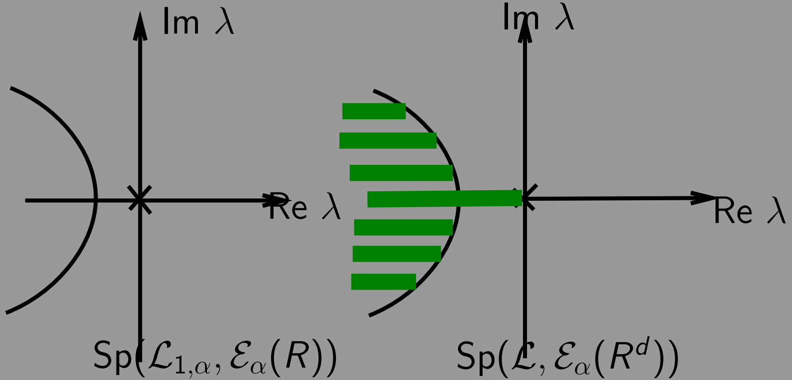
$$((\gamma_\alpha \otimes I_{H^k(\mathbb{R}^{d-1})})u)(z, y) = \gamma_\alpha(z) u(z, y), \qquad (z, y) \in \mathbb{R}^d$$


Im λ
Pass to the weighted space
Re λ
Im λ
$\mathrm{Sp}(\mathcal{L}_1, \mathcal{E}_0(R))$
$\mathrm{Sp}(\mathcal{L}_{1,\alpha}, \mathcal{E}_\alpha(R))$

If $\eta \in \mathrm{Sp}(\mathcal{L}_{1,\alpha})$, and $\lambda \in \mathrm{Sp}(\Delta_y)$, then $\eta + \lambda \in \mathrm{Sp}(\mathcal{L}_\alpha)$, where

$$(\mathcal{L}_\alpha u)(z,y) = \big(\mathcal{L}_{1,\alpha} u(\cdot\,,y)\big)(z) + \big(\Delta_y u(z,\,\cdot)\big)(y).$$

$(\mathcal{L}_{1,\alpha} w(z) \approx \eta w(z)$, $\Delta_y v(y) \approx \lambda v(y)$ then $u(z,y) = w(z)v(y)$
gives $\mathcal{L}_\alpha wz = (\mathcal{L}_{1,\alpha} w)v + w(\Delta_y v) \approx \eta wv + \lambda wv = (\eta+\lambda)wv)$



Hypotheses

There exists $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$ such that

(a) $\sup\{\mathrm{Re}\lambda : \lambda \in \mathsf{Sp}_{\mathsf{ess}}(\mathcal{L}_{1,\alpha})\} < 0$.

(b) The only element of $\mathsf{Sp}(\mathcal{L}_{1,\alpha})$ in $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq 0\}$ is a simple eigenvalue at $\lambda = 0$ with ϕ' being the respective eigenfunction.

(c) Under the assumption $f(u_1, 0) = (Au_1, 0)$ on the nonlinearity, we linearize $u_t = \Delta_x u + cu_z + f(u)$ at the left end state $(0, 0)$ and obtain

$$L^- = \begin{pmatrix} L^{(1)} & \partial_{u_2} f_1(0,0) \\ 0 & L^{(2)} \end{pmatrix} \tag{31}$$

$$L^{(1)} = \Delta_x + c\partial_z + \partial_{u_1} f_1(0,0) = \partial_{zz} + c\partial_z + A_1, \tag{32}$$

$$L^{(2)} = \Delta_x + c\partial_z + \partial_{u_2} f_2(0,0). \tag{33}$$

Hypotheses

$$L^{(1)} = \Delta_x + c\partial_z + \partial_{u_1} f_1(0,0) = \partial_{zz} + c\partial_z + A_1, \qquad (34)$$

$$L^{(2)} = \Delta_x + c\partial_z + \partial_{u_2} f_2(0,0). \qquad (35)$$

(1) The operator $\mathcal{L}^{(1)}$ on $\mathcal{E}_0^{n_1}$ induced by (34) generates a bounded semigroup, that is, $\|e^{t\mathcal{L}^{(1)}}\|_{\mathcal{B}(\mathcal{E}_0)} \le K$ for some $K > 0$ and all $t \ge 0$;

(2) The operator $\mathcal{L}^{(2)}$ on $\mathcal{E}_0$ induced by (35) satisfies

$$\sup\{\mathrm{Re}\lambda : \lambda \in \mathrm{Sp}(\mathcal{L}^{(2)})\} < 0,$$

so that there exist numbers $\rho > 0$ and $K > 0$, for which the inequality

$$\|e^{t\mathcal{L}^{(2)}}\|_{\mathcal{B}(\mathcal{E}_0)} \le Ke^{-\rho t}$$

holds for all $t \ge 0$.

We also define the projection operators

$$(\mathcal{P}^c u)(z,y) = \big(P^c u(\cdot, y)\big)(z), \quad (\mathcal{P}^s u)(z,y) = \big(P^s u(\cdot, y)\big)(z).$$

$$\begin{aligned} &\|e^{t\mathcal{L}_{1,\alpha}} P^s\|_{\mathcal{B}(H^k_\alpha(\mathbb{R}))} \le Ce^{-\nu t}; \\ &\|e^{t\mathcal{L}_\alpha} \mathcal{P}^s\|_{\mathcal{B}(\mathcal{E}_\alpha)} \le Ce^{-\nu t}; \\ &\|e^{t\mathcal{L}_\mathcal{E}}\|_{\mathcal{B}(\mathcal{E})} \le C. \end{aligned}$$

Here, $\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ and $\|\cdot\|_\mathcal{E} = \max\{\|\cdot\|_{\mathcal{E}_0}, ||\cdot||_{\mathcal{E}_\alpha}\}$. Also, the semigroup $S_{\Delta_y}(t)$ generated by the linear operator Δ_y for all $t \ge 0$ satisfies the following decay estimates with some $\beta > 0$:

(a) $||S_{\Delta_y}(t)u||_{H^k(\mathbb{R}^{d-1})} \le C||u||_{H^k(\mathbb{R}^{d-1})}$,

(b) $||S_{\Delta_y}(t)u||_{H^k(\mathbb{R}^{d-1})} \le$
$C(1+t)^{-(d-1)/4}||u||_{L^1(\mathbb{R}^{d-1})} + Ce^{-\beta t}||u||_{H^k(\mathbb{R}^{d-1})}$.

We study solutions u of $u_t = \Delta_x u + cu_z + f(u)$ near ϕ such that $u = \phi + \mathit{small}$. We seek a function $v(\cdot)$ with small v^0 such that

$$u(z,y) = \phi(z) + \mathit{small} = \phi(z - q(y)) + v(z,y), \quad (z,y) \in \mathbb{R}^d. \tag{36}$$

Substituting $u = \phi_q + v$ into $u_t = \Delta_x u + cu_z + f(u)$ we obtain, as in [Kapitula], a system of equations

$$\begin{aligned} \partial_t v - \phi_q' \partial_t q = Lv + (df(\phi_q) - df(\phi))v + N(\phi_q, v)v \\ - \Delta_y q \phi_q' + (\nabla_y q \cdot \nabla_y q)\phi_q'', \end{aligned} \tag{37}$$

use $\mathcal{P}^s$ and $\mathcal{P}^c$ to uncouple (37) we obtain a system of equations:

$$\partial_t v = Lv + F_1(v,q),\ \partial_t q = \Delta_y q + F_2(v,q).$$

$$\partial_t v = Lv + F_1(v,q),\ \partial_t q = \Delta_y q + F_2(v,q).$$

Here, $F_i(v,q)$, $i = 1,2$ defines locally Lipschitz mappings on $\mathcal{E}$ and $H^k(\mathbb{R}^{d-1})$, respectively, and satisfy the estimates:

(a) $\|F_1(v,q)\|_0 \leq C_K(\|v\|_0\|v\|_\alpha + \|v\|_0\|v_2\|_0 + \|q\|_{H^k}\|v\|_\alpha + \|\nabla_y q\|_{H^k}^2)$,

(b) $||F_1(v,q)||_\alpha \leq C_K\big(||v||_0||v||_\alpha + ||q||_{H^k}||v||_\alpha + ||\nabla_y q||_{H^k}^2\big)$,

(c) $||F_2(v,q)||_{H^k(\mathbb{R}^{d-1})} \leq C_K\big(||v||_0||v||_\alpha + ||q||_{H^k}||v||_\alpha + ||\nabla_y q||_{H^k}^2\big)$,

(d) $||F_2(v,q)||_{L^1(\mathbb{R}^{d-1})} \leq C_K\big(||v||_0||v||_\alpha + ||q||_{H^k}||v||_\alpha + ||\nabla_y q||_{H^k}^2\big)$.

Theorem

Assume $k \geq [\frac{d+1}{2}]$. *There exist a small* $\delta_0 > 0$ *and a constant* $C > 0$ *such that for each* $0 < \delta < \delta_0$ *there exists* $0 < \eta < \delta$ *such that the following is true. Let* $(v^0, q^0) \in \mathcal{E}^n \times H^k(\mathbb{R}^{d-1})$ *be the initial condition satisfying*
$E_k = ||v^0||_{\mathcal{E}} + \|q^0\|_{H^{k+1}(\mathbb{R}^{d-1})} + \|q^0\|_{W^{1,1}(\mathbb{R}^{d-1})} \leq \eta$ *and let* $(v(t), q(t)) \in \mathcal{E}^n \times H^k(\mathbb{R}^{d-1})$ *be the solution to* $\partial_t v = Lv + F_1(v, q)$, $\partial_t q = \Delta_y q + F_2(v, q)$ *with the initial condition* (v^0, q^0). *Then for all* $t > 0$,

(1) $(v(t), q(t))$ *is defined in* $\mathcal{E}^n \times H^k(\mathbb{R}^{d-1})$;

(2) $||v(t)||_{\mathcal{E}} + ||q(t)||_{H^k} \leq \delta$;

(3) $||v(t)||_{\alpha} \leq C(1+t)^{-(d+1)/2} E_k$;

(4) $||q(t)||_{H^k} \leq C(1+t)^{-(d-1)/4} E_k$;

(5) $||v_1(t)||_0 \leq CE_k$;

(6) $||v_2(t)||_0 \leq C(1+t)^{-(d+1)/2} E_k$.

Thank you!!