Reaction-diffusion equations in biomedical applications

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Reaction-diffusion equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u) \]

- \( u(x,t) \) – depending on considered applications describes temperature or concentration distribution, the density of some populations, etc.
- Heat or mass production or reproduction rate
- Mass diffusion or heat conduction or random motion of individuals

One equation or systems of equations
Beginning of the theory of reaction-diffusion equations

- Heat explosion (Semenov, Frank-Kamenetskii, 1930s)
- Wave propagation (Fisher, KPP, Zeldovich-Frank-Kamenetskii, 1938)
- Pattern formation (Turing, 1952)
Frank-Kamenetskii model of heat explosion

\[ \frac{\partial u}{\partial t} = \Delta u + e^u \]

Temperature distribution

Existence of stationary solutions or blow-up (unbounded) solution?

Ammonium nitrate explosion, Beyrouth, 2020
Wave propagation (KPP-Fisher equation)

\[ \frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + F(u) \]

\[ u(x,t) = w(x-ct) \]

\[ F(u) = u(1-u) \]
Ecological invasions

Invasion of American muskrat in Europe

Skellam, 1951
Epidemic progression

Infection spreading in the USA
Combustion engines, fires
Tumor growth

\[ \frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + au(1 - u) \]
Excitable medium: brain, heart

Nerve impulse: Hodgkin-Huxley model
(from Keener, Sneyd)

Belousov, Zhabotinskii reactions

Heart waves
(from D. Noble paper)
Pattern formation: Turing structures

\[ \frac{\partial u}{\partial t} = d_u \frac{\partial^2 u}{\partial x^2} + F(u,v) \]

\[ \frac{\partial v}{\partial t} = d_v \frac{\partial^2 v}{\partial x^2} + G(u,v) \]
Existence and stability of waves and pulses
Main definitions

Travelling wave solution:

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u) \]

\[ u(x, t) = w(x - ct) \]

\[ c = \text{wave speed – unknown constant} \]

satisfies the problem:

\[ w'' + cw' + F(w) = 0 \]

\[ w(-\infty) = 1, \quad w(\infty) = 0 \]

Three types of nonlinearity:

- Monostable
- Bistable
- Unstable
Wave existence: monostable case

First-order system:

\[ w'' + cw' + F(w) = 0 \]

\[ w(-\infty) = 1, \quad w(\infty) = 0 \]

We look for a trajectory connecting (1,0) and (0,0)
Stationary points

\[ (0, 0) \] Linearized system

\[
\begin{align*}
\begin{cases} 
    w' = p \\
    p' = -c p - F'(0) w
\end{cases}
\end{align*}
\]

\[
A = \begin{pmatrix} 0 & 1 \\ -F'(0) & -c \end{pmatrix}
\]

\[ F'(0) > 0 \]

Eigenvalues

\[
A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ -F'(0) & -c - \lambda \end{pmatrix}
\]

\[
\det(A - \lambda I) = 0 \quad \Rightarrow \\
\lambda^2 + c \lambda + F'(0) = 0
\]

\[
\lambda = -\frac{c}{2} \pm \frac{\sqrt{c^2 - 4 F'(0)}}{2}
\]

\[ \text{if } c > 2 \sqrt{F'(0)} \quad \text{stable node (equal?)} \]
\[ \text{if } c < -2 \sqrt{F'(0)} \quad \text{unstable node} \]
\[ c^2 < 4 F'(0) \quad \text{focus} \]

\( -3 - \)
Wave existence: monostable case

First-order system:

\[ w'' + cw' + F(w) = 0 \]

\[ w(-\infty) = 1, \ w(\infty) = 0 \]

We look for a trajectory connecting (1,0) and (0,0)

Theorem 1. Monotone waves exist for all values of the speed \( c \) greater than or equal to some minimal speed \( c_0 \)
Wave existence: bistable case

First-order system:

\[ w'' + cw' + F(w) = 0 \]

\[ w(-\infty) = 1, \quad w(\infty) = 0 \]

\[ w' = p, \quad p' = -cp - F(w) \]

We look for a trajectory connecting (1,0) and (0,0)

Theorem 2. A monotone waves exists for a single value of speed c.
Examples

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u) \]

Population dynamics

\[ F(u) = au^k(1 - u) - \sigma u \]

Combustion

\[ F(u) = ae^{zu}(1 - u) - \sigma u \]

\( k = 1 \)

\( k = 2 \)
Existence of pulses

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u) \]

Pulse – positive stationary solution with 0 limits at infinity

Bistable nonlinearity

\[ w'' + F(w) = 0 \]

\[ w(\pm \infty) = 0 \]

\[ w' = p, \quad p' = -F(w) \]

Theorem 3. Pulse exists if and only if

\[ \int_0^1 F(u)\,du > 0 \]
Examples

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u) \]

**Local model**

\[ F(u) = au^k(1 - u) - \sigma u \]

\( k = 1 \)  
No pulse

\( k = 2 \)

**Nonlocal model**

\[ F(u) = au^2 \left( 1 - \int_{-\infty}^{\infty} u(y) dy \right) - \sigma u \]

\[ \frac{r}{a} \left( 1 - \int_{-\infty}^{\infty} u(y) dy \right) \]

\[ u'' + ru^2 - \sigma u = 0 \]

\( r = a \left( 1 - \int_{-\infty}^{\infty} u(y) dy \right) \)

two pulses
Generalizations

- Systems of equations
- Multi-dimensional equations and systems
- Nonlocal (integrodifferential equations)
- Delay equations
Stability of waves: definitions

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u)
\]

\[u(x, t) = w(x - ct)\]

\[w'' + cw' + F(w) = 0\]

\[w(-\infty) = 1, \quad w(\infty) = 0\]

Invariance with respect to translation: \(w(x+h)\) is a solution for any real \(h\)
Stability of waves: definitions

Wave $w(x)$ is a solution of the equation

$$w'' + cw' + F(w) = 0$$

Linearized operator

$$Lw = v'' + cv' + F'(w(x))v$$

Existence of zero eigenvalue: $Lw' = 0 \rightarrow$ the previous stability result is not applicable

**Stability with shift**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u)$$

$$u(x, 0) = w(x) + \phi(x)$$

Initial condition

Asymptotic stability with shift (with respect to small perturbations): for any sufficiently small perturbation, solution $u(x,t)$ converges to the shifted wave $w(x-ct+h)$ for some $h$. 
Krein-Rutman theorem

Consider a linear elliptic operator in a bounded domain:

\[ Lu = \Delta u + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} + b(x)u \quad u|_{\partial \Omega} = 0 \]

The principal eigenvalue (with the maximal real part) of the problem

\[ Lu = \lambda u \]

is real, simple, and the corresponding eigenfunction is positive.

There are no other positive eigenfunctions.

Stability of waves (scalar equation)

Equation for the wave \( w(x) \)

\[
\frac{d^2 w}{dx^2} + cw' + F(w) = 0
\]  

(1)

Differentiating (1)

\[
\frac{d^2 u}{dx^2} + cu' + F'(w(x))u = 0
\]  

(2)

Linearized operator

\[
Lu = \frac{d^2 u}{dx^2} + cu' + F'(w(x))u
\]

From (2): operator \( L \) has 0 eigenvalue and the corresponding eigenfunction \( w'(x) \) is positive. Hence \( 0 \) is the eigenvalue with the maximal real part, and all other eigenvalues lie in the left half-plane.

**Theorem.** Monotone waves for the scalar equation are stable.
Wave speed: minimax representation

\[ w'' + cw' + F(w) = 0 \]

Minimax representation

\[ c = \frac{w'' + F(w)}{-w'} \]

Test function

\[ \inf_{\rho'} \frac{\rho'' + F(\rho)}{-\rho'} \leq c \leq \sup_{\rho'} \frac{\rho'' + F(\rho)}{-\rho'} \]

The proof uses global asymptotic stability of waves for monotone systems.
Instability of waves and pulses

Equation for the wave $w(x)$

\[ w'' + cw' + F(w) = 0 \]  

(1)

Differentiating (1)

\[ u = w'(x) \]

\[ u'' + cu' + F'(w(x))u = 0 \]

(2)

Linearized operator

\[ Lu = u'' + cu' + F'(w(x))u \]

From (2): operator L has 0 eigenvalue and the corresponding eigenfunction $w'(x)$ has variable sign. Hence 0 is not the eigenvalue with the maximal real part, and there are eigenvalues in the right half-plane.

Theorem. Non-monotone waves and pulses for the scalar equation are unstable.
Reaction-diffusion systems: monotone systems

\[ \frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + F(u) \]

\[ \frac{\partial F_i}{\partial u_j} \geq 0, \quad i \neq j \]

Bistable: existence, uniqueness, convergence to waves, minimax representation of the speed

Monostable: existence for \( c \geq c_0 \), stability, minimax representation of the minimal wave speed

Unstable: non-existence